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Research Article

Boundary value Problems for nonlinear implicit fractional differential equations

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Abstract

The aim of this paper is to study the existence and uniqueness of solutions for the boundary value problem for nonlinear implicit fractional differential equations involving standard Riemann–Liouville fractional derivative. Our results are based on the Banach's contraction mapping principal and Krasnoselskii's fixed point theorem. Finally, one illustrative example is given to demonstrate the obtained results.

Keywords: Riemann–Liouville fractional derivative, Implicit fractional differential equations, Fractional integral, Boundary value problem, Krasnoselskii's fixed point theorem.

1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The idea of an arbitrary order derivative was first seen in the letter written to Leibniz by L'Hospital in 1695. In recent years, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of applied sciences and engineering such as acoustic control, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, proteins, electromagnetics, optics, medicine, economics, astrophysics, chemical engineering, chaotic dynamics, statistical physics and so on. see for example ([1],[3],[5],[6],[12],[13],[14],[15],[19],[21]), and the references therein. Due to its importance in different fields, it is receiving increasing attention and has held a central place in attention researchers and mathematicians.

Boundary value problems and initial value problems for fractional differential equation take place in many areas of physics and applied mathematics such as heat conduction, underground water flow, chemical engineering, thermoelasticity, and plasma physics. Recently, many authors have investigated the existence of solutions of boundary value problem and initial value problem for fractional differential equations with Caputo fractional derivative and Riemman fractional derivative. In survey paper [2], Agarwal et al. established sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions involving the Caputo fractional derivative. In [4], Bai investigated the existence and uniqueness of positive solutions for a nonlocal boundary value problem of fractional differential equation with Caputo fractional derivative. Cui et al. [11] studied uniqueness for boundary value problem of fractional differential equation involving

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the Caputo fractional derivative. In [16], Nanware studied existence of solutions for a class of m-point boundary value problem of fractional differential equation involving the Caputo fractional derivative. Nanware and Dhaigude [17] established sufficient conditions for the existence and uniqueness of solutions for a class of boundary value problem for fractional differential equations. Su and Liu [20] obtained existence results for solution of boundary value problem of a nonlinear fractional differential equation. In [22], Tate and Dinde studied existence and interval of existence of solutions, uniqueness and other properties of a Cauchy problem for nonlinear fractional differential equations with constant coefficient involving the Caputo fractional derivative. Yan et al . [23] studied existence and uniqueness of solutions for fractional differential equations with nonlocal boundary conditions. In [24], Zhang studied existence and multiplicity of positive solutions for the nonlinear fractional differential equation boundary value problem.

In [7], Benchohra et al. established sufficient conditions for the existence of solutions for a boundary value problem for fractional differential equations of the type:

$$^{c}D^{\alpha}x(t) = f(t,x(t)),\ t \in [0,T],\ 0 < \alpha < 1$$

$$ax(0) + bx(T) = c$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a,\ b,\ c,$ are real constant with $a + b \neq 0$, and

$$^{c}D^{\alpha}x(t) = f(t,x(t)),\ t \in J := [0,T],\ T > 0,$$

$$x(0) + g(x) = x_0$$

where $g : C([0,T],\mathbb{R}) \to \mathbb{R}$ is a continuous function and x_0 is a real constant.

In [8], using Banach contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type, M. Benchohra and J.E. Lazreg investigated the existence and uniqness results for nonlinear implicit fractional differential equations (NIFDEs) with boundary conditions of the type:

$$^{c}D^{\alpha}y(t) = f(t,y(t),{}^{c}D^{\alpha}y(t)),\ t \in J = [0,T],\ T > 0,\ 1 < \alpha \leq 2$$

$$y(0) = y_0,\ y(T) = y_1$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $y_0, y_1 \in \mathbb{R}$.

In [9], Benchohra et al. studied the existence and uniqueness of solutions for boundary value problems with fractional order differential equations and non-linear integral conditions of the form

$$^{c}D^{\alpha}x(t) = f(t,x(t)),\ t \in J := [0,T],\ 1 < \alpha \leq 2,$$

$$x(0) - x'(0) = \int_0^T g(s,x)ds,$$

$$x(T) + x'(T) = \int_0^T h(s,x)ds,$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, and $f,\ g$ and $h : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In [10], using Krasnoselskii's fixed point theorem and Banach contraction principle, Benlabbes et al. studied the existence and uniqueness of solutions for the boundary value problem for a nonlinear fractional differential equation of the type:

$$D^{\alpha}_{0+}x(t) = f(t,x(t)),\ t \in J = [0,1],\ 2 < \alpha \leq 3$$

$$D^{\alpha-1}_{0+}x(0) = 0,\ D^{\alpha-2}_{0+}x(1) = 0,\ x(1) = 0,$$

where f is a given function and D^{α}_{0+} is the standard Riemman fractional derivative operator of order α.

In [18], using Krasnoselskii's fixed point theorem and Banach contraction principle, G. M. N'Guérékata investigated the existence and uniqueness of solutions to the Cauchy problem for the fractional differential equation with non local conditions of the type :

$$^{c}D^{\alpha}y(t) = f(t, y(t)),\ t \in I = [0, T],\ 0 < \alpha < 1$$

$$y(0) + g(y) = y_0$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a given continuous function and $g : C(I, \mathbb{X}) \to \mathbb{X}$ is continuous.

Motivated by the above cited works, in this paper, we investigate the existence and uniqueness results for the following problem of NIFDE:

$$D^{\alpha}_{0^+}x(t) = f(t, x(t), D^{\alpha}_{0^+}x(t)),\ t \in J = [0, 1],\ 2 < \alpha \leq 3 \tag{1.1}$$

$$D^{\alpha-1}_{0^+}x(0) = 0,\ D^{\alpha-2}_{0^+}x(1) = 0,\ x(1) = 0 \tag{1.2}$$

where $D^{\alpha}_{0^+}$ is the standard Riemann– Liouville fractional derivative of order α, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.
The paper is organized as follows. In Section 2, we present notations and definition used throughout the paper. In Section 3, we discuss the existence results for NIFDE (1.1)–(1.2) by using Banach contraction mapping principal and Krasnoselskii's fixed point theorem.

2 Preliminaries and notations

In this section, we collect some definitions, notations and results from ([14], [19]) which are used throughout this paper. By C(J, $\mathbb{R}$) we denote the Banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$||y||_{\infty} := sup\{|y(t)| : t \in J\}.$$

Definition 2.1. *The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $h \in L^1([a, b], \mathbb{R}_+)$ is defined by*

$$I^{\alpha}_{a}h(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} h(s)\, ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t}\, dt$.
when $a = 0$, we write $I^{\alpha}_{0}h(t) = (h * \varphi_{\alpha})(t)$, where $\varphi_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_{\alpha}(t) = 0$ for $t < 0$, and $\varphi_{\alpha} \to \delta$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2. *For a function h given on the interval $[0, \infty)$, the expression*

$$D^{\alpha}_{0^+}h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} (t-s)^{n-\alpha-1} h(s)\, ds,$$

is called the Riemann–Liouville fractional derivative of order α, where n=[α]+1 and [α] denotes the integer part of the real number α.

Remark 2.1.

i) For $\alpha < 0$, we use the convention that $D^{\alpha}_{0^+}h(t) = I^{-\alpha}_{0}h(t)$, for $t \geq 0$.

ii) For $\beta \in [0, \alpha)$, we have $D^{\alpha}_{0^+}I^{\beta}_{0^+}h = I^{\alpha-\beta}_{0^+}h$, with $D^{\alpha}_{0^+}I^{\alpha}_{0^+}h = h$.

iii) For $\lambda > -1,\ \lambda \neq \alpha-1, \alpha-2, ..., \alpha-n$, we have, for $t \geq 0$,

$$D^{\alpha}_{0^+} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$$

and

$$D^{\alpha}_{0^+} t^{\alpha-i} = 0,\ \forall i = 1, 2, ..., n.$$

Lemma 2.1. *[14] Let $\alpha > 0$. If $h \in C(J, \mathbb{R}) \cap L^1(J, \mathbb{R})$, then the fractional differential equation*

$$D^{\alpha}_{0^+} h(t) = 0$$

has a solutions $h(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}$, where $t \in J$, $c_i \in \mathbb{R}$, i=0,1 ,2 ,......,n are constants and n=[α]+1.

Lemma 2.2. *[14] Let $n-1 < \alpha < n$ and $h \in C(J, \mathbb{R})$. We have*

$$I^{\alpha}_{0^+} D^{\alpha}_{0^+} h(t) = h(t) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - .... - c_n t^{\alpha-n},\ t \in J,$$

where $c_i \in \mathbb{R}$, i=0,1 ,2 ,......,n are constants.

The following are the fundamental tools on which the proofs of our main results are based.

Definition 2.3. *Let $(\mathbb{X}, ||.||)$ be a normed space. A contraction of $\mathbb{X}$ is a mapping $T : \mathbb{X} \to \mathbb{X}$ that satisfies*

$$\forall x_1, x_2 \in \mathbb{X} : ||T(x_1) - T(x_2)|| \leq \beta\, ||x_1 - x_2||$$

for some real number $\beta < 1$.

Theorem 2.1. *(Banach fixed point theorem) Every contraction mapping on a complete metric space has a unique fixed point.*

Theorem 2.2. *(Krasnoselskii's fixed point theorem). Let $\mathscr{B}$ be a closed convex and nonempty subset of a Banach space $\mathbb{X}$. Let T_1, T_2 be two operators such that*
(i) $T_1 x + T_2 y \in \mathscr{B}$ whenever $x, y \in \mathscr{B}$;
(ii) T_2 is a contraction mapping.
(iii) T_1 is compact and continuous;
Then there exists $z \in \mathscr{B}$ such that $z = T_1 z + T_2 z$.

3 Existence of Solutions

Definition 3.1. *A function $x \in C(J, \mathbb{R})$ is said to be a solution to problem* (1.1)–(1.2) *if x satisfies the NIFDE* (1.1) *and the conditions* (1.2) *on J.*

Lemma 3.1. *[10] Let $2 < \alpha < 3$ and let $h : J \to \mathbf{R}$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds$$

if and only if x is a solution of the initial value problem for the fractional differential equation

$$D^{\alpha}_{0^+} x(t) = h(t),\ t \in J,$$

$$x(0) = x_0.$$

Lemma 3.2. *[10] Let $2<\alpha<3$ and let $h:J\to\mathbf{R}$ be a continuous function. A function x is a solution of problem*

$$D^{\alpha}_{0^+}x(t)=h(t),\ t\in J,$$

$$D^{\alpha-1}_{0^+}x(0)=0,\ D^{\alpha-2}_{0^+}x(1)=0,\ x(1)=0$$

if and only if x is a solution to the fractional integral equation

$$x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\,ds+\frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1(1-s)h(s)\,ds-\frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}h(s)\,ds$$

As a consequence of Lemma (3.2), we have the following result which is useful in our main results.

Lemma 3.3. *(See [10] for more details) Let $2<\alpha<3$ and let $f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. A function x is a solution of NIFDE* (1.1)–(1.2) *if and only if x is a solution to the fractional integral equation*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s),D^{\alpha}_{0^+}x(s))\,ds \\ &+ \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1(1-s)f(s,x(s),D^{\alpha}_{0^+}x(s))\,ds \\ &- \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,x(s),D^{\alpha}_{0^+}x(s))\,ds \end{aligned} \tag{3.3}$$

We investigate in our paper the NIFDE (1.1)–(1.2) with the following assumptions:

(H1): $f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

(H2): $|f(t,x,y)-f(t,\bar{x},\bar{y})|\le p(t)\,|x-\bar{x}|+N\,|y-\bar{y}|$, $t\in J$ and $x,y,\bar{x},\bar{y}\in\mathbb{R}$, where $p(t)\in C(J,\mathbb{R}_+)$, $0<N<1$.

(H3): $|f(t,x,y)|\le q(t)\,|x|+L\,|y|$, $t\in J$ and $x,y\in\mathbb{R}$, where $q(t)\in C(J,\mathbb{R}_+)$ and $0<L<1$.

Theorem 3.1. *Under assumptions (H1)-(H2), if*

$$\frac{P^*}{1-N}\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right)<1 \tag{3.4}$$

where $P^*=Sup\{p(t):t\in J\}$ and $0<N<1$, then Eq.(1.1)*)–*(1.2) *has a unique solution on J.*

Proof. We transform problem (1.1))–(1.2) into a fixed point problem. For this, consider the operator $T:C(J,\mathbb{R})\to C(J,\mathbb{R})$ defined by

$$\begin{aligned} Tx(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s),D^{\alpha}_{0^+}x(s))\,ds \\ &+ \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1(1-s)f(s,x(s),D^{\alpha}_{0^+}x(s))\,ds \\ &- \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,x(s),D^{\alpha}_{0^+}x(s))\,ds \end{aligned} \tag{3.5}$$

Clearly, the fixed points of the operator T are solution of the problem (1.1)–(1.2). Now, according to Theorem (2.1), it is enough to prove that T is a contraction. Let $x,y\in C(J,\mathbb{R})$ and for any $t\in J$, then we have

$$\begin{aligned} |Tx(t)-Ty(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f(s,x(s),D^{\alpha}_{0^+}x(s))-f(s,y(s),D^{\alpha}_{0^+}y(s))\right|ds \\ &+ \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1(1-s)\left|f(s,x(s),D^{\alpha}_{0^+}x(s))-f(s,y(s),D^{\alpha}_{0^+}y(s))\right|ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left|f(s,x(s),D^{\alpha}_{0^+}x(s))-f(s,y(s),D^{\alpha}_{0^+}y(s))\right|ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{p(s)\,|x(s)-y(s)|+N\left|D_{0^+}^{\alpha}x(s)-D_{0^+}^{\alpha}y(s)\right|\}\,ds \\
&+ \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{p(s)\,|x(s)-y(s)|+N\left|D_{0^+}^{\alpha}x(s)-D_{0^+}^{\alpha}y(s)\right|\}\,ds \\
&+ \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\{p(s)\,|x(s)-y(s)|+N\left|D_{0^+}^{\alpha}x(s)-D_{0^+}^{\alpha}y(s)\right|\}\,ds
\end{aligned}
$$

Note that, for any $t \in J$

$$
\begin{aligned}
\left|D_{0^+}^{\alpha}x(t)-D_{0^+}^{\alpha}y(t))\right| &\leq \left|f(t,x(t),D_{0^+}^{\alpha}x(t))-f(t,y(t),D_{0^+}^{\alpha}y(t)))\right| \\
&\leq p(t)\,|x(t)-y(t)|+N\left|D_{0^+}^{\alpha}x(t)-D_{0^+}^{\alpha}y(t)\right|
\end{aligned}
$$

This gives

$$
\left|D_{0^+}^{\alpha}x(t)-D_{0^+}^{\alpha}y(t)\right| \leq \frac{p(t)}{(1-N)}\,|x(t)-y(t)| \tag{3.6}
$$

Using (3.6), we have

$$
\begin{aligned}
|Tx(t)-Ty(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{\frac{p(s)}{(1-N)}\,|x(s)-y(s)|\}\,ds \\
&+ \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{\frac{p(s)}{(1-N)}\,|x(s)-y(s)|\}\,ds \\
&+ \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\{\frac{p(s)}{(1-N)}\,|x(s)-y(s)|\}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{P^*\,||x-y||_{\infty}}{(1-N)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds \\
&+ \frac{t^{\alpha-3}(1-t)P^*\,||x-y||_{\infty}}{(1-N)\Gamma(\alpha-1)}\int_0^1 (1-s)\,ds \\
&+ \frac{t^{\alpha-3}P^*\,||x-y||_{\infty}}{(1-N)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,ds \\
&\leq \frac{P^*t^{\alpha}\,||x-y||_{\infty}}{(1-N)\alpha\Gamma(\alpha)}+\frac{t^{\alpha-3}(1-t)P^*\,||x-y||_{\infty}}{2\Gamma(\alpha-1)(1-N)} \\
&+ \frac{t^{\alpha-3}P^*\,||x-y||_{\infty}}{(1-N)\alpha\Gamma(\alpha)} \\
&\leq \frac{P^*}{(1-N)}\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right)||x-y||_{\infty}
\end{aligned}
$$

from which we deduce that

$$
||Tx(t)-Ty(t)|| \leq \frac{P^*}{(1-N)}\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right)||x-y||_{\infty}
$$

Thus, T is a contraction due to the (3.4). As a consequence of Theorem (2.1), T has a unique fixed point which is a unique solution of problem (1.1)–(1.2). □

Our next result is based on Krasnoselskii's theorem.

Theorem 3.2. *Assume (H1)–(H3) and* $Q^* \leq \frac{(1-L)}{\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-1)}\right)}$, *where* $Q^* = Sup\{q(t) : t \in J\}$. *Then Eq.(1.1)-(1.2) has at least one solution on* J.

Proof. Consider $B_R := \{x \in C(J,R) : ||x||_\infty \leq R\}$. Now we define operators T_1 and T_2 on B_R as follows:

$$\begin{aligned} T_1x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),D^\alpha_{0^+}x(s))\,ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-s)f(s,x(s),D^\alpha_{0^+}x(s))\,ds \end{aligned}$$

and

$$\begin{aligned} T_2x(t) &= -\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)f(s,x(s),D^\alpha_{0^+}x(s))\,ds \\ &- \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,x(s),D^\alpha_{0^+}x(s))\,ds. \end{aligned}$$

We will prove $T_1x + T_2y \in B_R$, for any $x,y \in B_R$.

Let $x,y, \in B_R$ and $t \in J$, then we have

$$\begin{aligned} |T_1x(t)+T_2y(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \left|f(s,x(s),D^\alpha_{0^+}x(s))\right| ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\left|f(s,x(s),D^\alpha_{0^+}x(s))\right| ds \\ &+ \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\left|f(s,y(s),D^\alpha_{0^+}y(s))\right| ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f(s,y(s),D^\alpha_{0^+}y(s))\right| ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{q(s)\,|x(s)|+L\left|D^\alpha_{0^+}x(s)\right|\}|\,ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{q(s)\,|x(s)|+L\left|D^\alpha_{0^+}x(s)\right|\}\,ds \\ &+ \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{q(s)\,|y(s)|+L\left|D^\alpha_{0^+}y(s)\right|\}\,ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\{q(s)\,|y(s)|+L\left|D^\alpha_{0^+}y(s)\right|\}\,ds \end{aligned}$$

Note that, for any $t \in J$

$$\begin{aligned} \left|D^\alpha_{0^+}y(t)\right| &\leq \left|f(t,y(s),D^\alpha_{0^+}y(s))\right| \\ &\leq q(t)\,|y(t)|+L\left|D^\alpha_{0^+}y(t)\right| \end{aligned}$$

This gives

$$\left|D^\alpha_{0^+}y(t)\right| \leq \frac{q(t)}{(1-L)}\,|y(t)| \tag{3.7}$$

Using (3.7), we have

$$\begin{aligned} |T_1x(t)+T_2y(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{q(s)\,|x(s)|+L\frac{q(s)}{(1-L)}\,|x(s)|\}|\,ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{q(s)\,|x(s)|+L\frac{q(s)}{(1-L)}\,|x(s)|\}\}\,ds \end{aligned}$$

$$
\begin{aligned}
&+ \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{q(s)\,|y(s)| + L\frac{q(t)}{(1-L)}\,|y(s)|\}\,ds \\
&+ \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\{q(s)\,|y(s)| + L\frac{q(t)}{(1-L)}\,|y(s)|\}\,ds \\
&\leq \frac{Q^*R}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds + \frac{Q^*Rt^{\alpha-3}}{(1-L)\Gamma(\alpha-1)}\int_0^1 (1-s)\,ds \\
&+ \frac{Q^*Rt^{\alpha-2}}{(1-L)\Gamma(\alpha-1)}\int_0^1 (1-s)\,ds + \frac{Q^*Rt^{\alpha-3}}{(1-L)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,ds \\
&\leq \left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-1)}\right)\frac{Q^*R}{(1-L)} \\
&\leq R
\end{aligned}
$$

Thus

$$||T_1x(t) + T_2y(t)||_\infty \leq R.$$

This gives $T_1x + T_2y \in B_R$.
Now, we will prove that T_2 is a contraction mapping on $C(J,R)$. Let $x, y \in C(J,R)$ and $t \in J$. We have

$$
\begin{aligned}
|T_2x(t) - T_2y(t)| &\leq \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\left|f(s,x(s),D^\alpha_{0^+}x(s)) - f(s,y(s),D^\alpha_{0^+}y(s))\right|\,ds \\
&+ \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f(s,x(s),D^\alpha_{0^+}x(s)) - f(s,y(s),D^\alpha_{0^+}y(s))\right|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{p(s)\,|x(s)-y(s)| + N\left|D^\alpha_{0^+}x(s) - D^\alpha_{0^+}y(s)\right|\}\,ds \\
&+ \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\{p(s)\,|x(s)-y(s)| + N\left|D^\alpha_{0^+}x(s) - D^\alpha_{0^+}y(s)\right|\}\,ds.
\end{aligned}
$$

Using (3.6), we have

$$
\begin{aligned}
|T_2x(t) - T_2y(t)| &\leq \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{\frac{p(s)}{(1-N)}\,|x(s)-y(s)|\}\,ds \\
&+ \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\{\frac{p(s)}{(1-N)}\,|x(s)-y(s)|\}\,ds \\
&\leq \frac{t^{\alpha-2}P^*\,||x-y||_\infty}{(1-N)\Gamma(\alpha-1)}\int_0^1 (1-s)\,ds \\
&+ \frac{t^{\alpha-3}P^*\,||x-y||_\infty}{(1-N)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,ds \\
&\leq \frac{P^*\,||x-y||_\infty}{2(1-N)\Gamma(\alpha-1)} + \frac{P^*\,||x-y||_\infty}{(1-N)\Gamma(\alpha+1)} \\
&= \left(\frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)}\right)\frac{P^*}{(1-N)}\,||x-y||_\infty
\end{aligned}
$$

from which we deduce that

$$||T_2x(t) - T_2y(t)||_\infty \leq \left(\frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)}\right)\frac{P^*}{(1-N)}\,||x-y||_\infty$$

Using the condition (3.4), we conclude that T_2 is contraction. Since x is continuous, then $T_1x(t)$ is continuous in view of (H1).

Finally, we will prove the compactness of T_1. From the Ascoli–Arzela Theorem, it is sufficient to prove that for each bounded subset B_R of $C(J,R)$, the set TB_R is bounded and is equicontinuous. Let B_R be a bounded subset of $C(J,R)$. we prove that T_1B_R is a bounded subset of $C(J,R)$. Let $x \in B_R$ and $t \in J$. We have

$$\begin{aligned} |T_1x(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,x(s),D_{0^+}^{\alpha}x(s))\right| ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\left|f(s,x(s),D_{0^+}^{\alpha}x(s))\right| ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{q(s)\,|x(s)|+L\left|D_{0^+}^{\alpha}x(s)\right|\}\,ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{q(s)\,|x(s)|+L\left|D_{0^+}^{\alpha}x(s)\right|\}\,ds. \end{aligned}$$

Using (3.7), we have

$$\begin{aligned} |T_1x(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{q(s)\,|x(s)|+L\frac{q(s)}{(1-L)}\,|x(s)|\}\,ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{q(s)\,|x(s)|+L\frac{q(s)}{(1-L)}\,|x(s)|\}\,ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{\frac{q(s)}{(1-L)}\,|x(s)|\}|\,ds \\ &+ \frac{t^{\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\{\{\frac{q(s)}{(1-L)}\,|x(s)|\}\}\,ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{Q^*R}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds+\frac{Q^*Rt^{\alpha-3}}{(1-L)\Gamma(\alpha-1)}\int_0^1 (1-s)\,ds \\ &= \frac{Q^*R}{(1-L)\Gamma(\alpha+1)}t^{\alpha}+\frac{Q^*Rt^{\alpha-3}}{2(1-L)\Gamma(\alpha-1)} \\ &\leq \left(\frac{1}{2\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha+1)}\right)\frac{Q^*R}{(1-L)} \\ &\leq R. \end{aligned}$$

Now let's prove $T_1x(t)$ is equicontinuous.
Let $t_1,t_2 \in J$ with $t_1 < t_2$ and $x \in B_R$

$$\begin{aligned}
|T_1x(t_1)-T_1x(t_2)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}\left|f(t,x(s),D_{0^+}^{\alpha}x(s))\right|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|f(t,x(s),D_{0^+}^{\alpha}x(s))\right|ds \\
&+ \frac{(t_1^{\alpha-3}-t_2^{\alpha-3})}{\Gamma(\alpha-1)}\int_0^1(1-s)\left|f(s,x(s),D_{0^+}^{\alpha}x(s))\right|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}\{q(s)|x(s)|+L\left|D_{0^+}^{\alpha}x(s)\right|\}ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\{q(s)|x(s)|+L\left|D_{0^+}^{\alpha}x(s)\right|\}ds \\
&+ \frac{(t_1^{\alpha-3}-t_2^{\alpha-3})}{\Gamma(\alpha-1)}\int_0^1(1-s)\{q(s)|x(s)|+L\left|D_{0^+}^{\alpha}x(s)\right|\}ds
\end{aligned}$$

Using (3.7), we get

$$\begin{aligned}
|T_1x(t_1)-T_1x(t_2)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}\frac{q(s)}{(1-L)}|x(s)|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{q(s)}{(1-L)}|x(s)|ds \\
&+ \frac{(t_1^{\alpha-3}-t_2^{\alpha-3})}{\Gamma(\alpha-1)}\int_0^1(1-s)\frac{q(s)}{(1-L)}|x(s)|ds \\
&\leq \frac{Q^*R}{(1-L)\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}ds \\
&+ \frac{Q^*R}{(1-L)\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds+\frac{Q^*R(t_1^{\alpha-3}-t_2^{\alpha-3})}{(1-L)\Gamma(\alpha-1)}\int_0^1(1-s)ds \\
&\leq \frac{Q^*R}{(1-L)\Gamma(\alpha+1)}\{(t_2-t_1)^{\alpha}+t_1^{\alpha}-t_2^{\alpha}\} \\
&+\frac{Q^*R}{(1-L)\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}+\frac{Q^*R(t_1^{\alpha-3}-t_2^{\alpha-3})}{(1-L)2\Gamma(\alpha-1)} \\
&\leq \frac{Q^*R}{(1-L)\Gamma(\alpha+1)}\{2(t_2-t_1)^{\alpha}+t_1^{\alpha}-t_2^{\alpha}\}+\frac{Q^*R(t_1^{\alpha-3}-t_2^{\alpha-3})}{(1-L)2\Gamma(\alpha-1)},
\end{aligned}$$

from which we deduce the desired property. Finally, as a consequence of Krasnoselskii's fixed point theorem, we conclude that Eq. (1.1)–(1.2) has at least one solution. □

4 Illustrative Example

Example 4.1. *Consider the boundary value problem:*

$$D_{0^+}^{\frac{14}{5}}y(t)=\frac{e^{-t}}{(9+e^t)}\left[\frac{|y(t)|}{1+|y(t)|}\right]-\frac{1}{2}\left[\frac{\left|D_{0^+}^{\frac{14}{5}}y(t)\right|}{1+\left|D_{0^+}^{\frac{14}{5}}y(t)\right|}\right],t\in J=[0,1] \tag{4.8}$$

$$D_{0^+}^{\frac{9}{5}}y(0)=0,\ D_{0^+}^{\frac{4}{5}}y(1)=0,\ y(1)=0 \tag{4.9}$$

Set $f(t,x,y)=\frac{e^{-t}}{(9+e^{t})}[\frac{x}{1+x}]-\frac{1}{2}[\frac{y}{1+y}]$, $t\in[0,1]$, $x,y\in[0,\infty)$. *Clearly f is continuous. For each* $x,\bar{x},y,\bar{y}\in R$ *and* $t\in[0,1]$:

$$\begin{aligned}|f(t,x,y)-f(t,\bar{x},\bar{y})| &\leq \frac{e^{-t}}{(9+e^{t})}|x-\bar{x}|+\frac{1}{2}|y-\bar{y}|\\ &\leq \frac{1}{10}|x-\bar{x}|+\frac{1}{2}|y-\bar{y}|.\end{aligned}$$

we see that $p(t)=\frac{e^{-t}}{(9+e^{t})}:[0,1]\to(0,\infty)$ *is continuous function. Hence condition (H2) is satisfied with* $P^{*}=\frac{1}{10}$, $N=\frac{1}{2}$.

We have $\frac{P^{*}}{1-N}(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)})=\frac{\frac{1}{10}}{1-\frac{1}{2}}(\frac{2}{\Gamma(\frac{14}{5}+1)}+\frac{1}{2\Gamma(\frac{14}{5}-1)})=0.192579<1$

Also for each $x,y\in R$ *and* $t\in[0,1]$:

$$\begin{aligned}|f(t,x,y)| &\leq \frac{e^{-t}}{(9+e^{t})}|x|+\frac{1}{2}|y|\\ &\leq \frac{1}{10}|x|+\frac{1}{2}|y|.\end{aligned}$$

we see that $q(t)=\frac{e^{-t}}{(9+e^{t})}:[0,1]\to(0,\infty)$ *is continuous function.*

Hence the condition (H3) is satisfied with $Q^{*}=\frac{1}{10}$, $L=\frac{1}{2}$. *We have* $\frac{(1-L)}{\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-1)}\right)}=0.333393>Q^{*}=\frac{1}{10}$.

It follows from Theorem (3.1)–(3.2) *that the problem (4.8)–(4.9) has unique solution on [0,1].*

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