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Ulam Stabilities for a class of nonlinear mixed fractional integro–differential equations

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Abstract. The aim of this study is to investigate stabilities of Ulam–Hyers, Ulam–Hyers–Rassias and semi–Ulam–Hyers-Rassias on closed interval $[a,b]$ for a class of nonlinear mixed fractional integro–differential equation involving ψ–Hilfer fractional derivative. Banach fixed point theorem is used to obtain our main results.

1 Introduction

The branch of fractional calculus is as old as the differential calculus. In recent years, the fractional calculus has received increasing attention within the scientific community and plays an important role in many areas such as engineering, mathematics, physics and bio-engineering and so on. (see [4, 10, 18, 23] and the references therein).

There are many definitions of fractional integrals and derivatives within the literature, however the foremost popular definitions are the Caputo and Riemann–Liouville. Hilfer has introduced a generalized type of the Riemann–Liouville fractional derivative. In short, Hilfer fractional derivative is an interpolation between the Riemann–Liouville and Caputo fractional derivatives. For more details, see [6, 7, 8, 9, 10, 29, 30, 32] and the references therein.

Recently, many researchers have studied the existence and uniqueness of the nonlinear fractional differential and integro–differential equations and obtained many interesting results by using all kinds of fixed point theorems, for example, by Cabrera et al. [2], Jagtap and Kharat [11], Kharat [12], Kharat et al. [13], Kendre and Kharat [15], Kendre et al. [14, 16, 17], N'Guérékata [19], Pierri and O'Regan [22], Ren et al. [24], Wang and Li [31] and the references therein.

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Tate et al.[27], studied the existence and interval of existence of solutions, uniqueness, continuous dependence of solutions on initial conditions, estimates on solutions and continuous dependence on parameters and functions involved in nonlinear fractional integro–differential equations with constant coefficient. In [26], by using generalized Banach fixed point theorem, Tate et al. discussed the existence and interval of existence of solutions, uniqueness, continuous dependence of solutions on initial conditions, estimates on solutions and continuous dependence on parameters and functions involved in the nonlinear fractional differential equation with constant coefficient involving Caputo fractional derivative.

The study of Ulam–Hyers stability and Ulam–Hyers–Rassias stability has become the topic of interest of many mathematicians. Detailed discussions regarding the Ulam-type stability have been made in [1, 3, 21, 28, 30, 31]. Recently, Vanterler and Oliveira [21], using ψ–Hilfer fractional derivative and the Banach fixed point theorem, investigated the stabilities of Ulam–Hyers, Ulam–Hyers–Rassias and semi–Ulam–Hyers–Rassias for a class of fractional integro–differential equation.

The above results motivate us and therefore, in this paper, we obtain Ulam–Hyers, Ulam–Hyers–Rassias and semi–Ulam–Hyers–Rassias in the interval $[a,b]$ for the class of nonlinear mixed fractional integro–differential equations (NMFIDEs)

$$\begin{cases} {}^{H}D_{+a}^{\alpha,\beta;\psi}y(x) = f\left(x, y(x), \int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau, \int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right), \\ I_{a+}^{1-\gamma;\psi}y(a) = c, \ \gamma = \alpha+\beta(1-\alpha) \end{cases} \tag{1.1}$$

where ${}^{H}D_{+a}^{\alpha,\beta;\psi}$ denotes the ψ–Hilfer fractional derivative, $I_{a+}^{1-\gamma;\psi}(.)$ is ψ– Riemann–Liouville integral, with $0<\alpha<1$, $0<\beta\leq 1$, $y\in C^1[a,b]$, for $x\in[a,b]$, a and b are fixed real numbers, $f:[a,b]\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$, $K:[a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ and $H:[a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ are given continuous functions and $\delta:[a,b]\to[a,b]$ is a continuous delay function.

The rest of the paper is organized as follows. In Section 2, some definitions, notations and basic results are given. Section 3 is devoted to study the Ulam–Hyers–Rassias stability of the NMFIDE (1.1). Semi–Ulam–Hyers–Rassias and Ulam–Hyers stability of the NMFIDE (1.1) included in Section 4 .

2 Preliminaries

Let $[a,b]$ $(0<a<b<\infty)$ be a finite interval on the half– axis $\mathbb{R}^+$ and let $C[a,b]$ be the space of continuous function f on $[a,b]$ with the norm

$$||f||_{C[a,b]} = \max_{x\in[a,b]} |f|. \tag{2.1}$$

The weighted space $C_{1-\gamma;\psi}[a,b]$ of continuous f on $(a,b]$ is defined by

$$C_{1-\gamma;\psi}[a,b] = \{f:(a,b]\to\mathbb{R}; (\psi(x)-\psi(a))^{1-\gamma}f(x)\in C[a,b]\}, \tag{2.2}$$

$0\leq\gamma<1$ with the norm

$$||f||_{C_{1-\gamma;\psi}[a,b]} = \left|\left|(\psi(x)-\psi(a))^{1-\gamma}f(x)\right|\right|_{C[a,b]} = \max_{x\in[a,b]}\left|(\psi(x)-\psi(a))^{1-\gamma}f(x)\right|. \tag{2.3}$$

The weighted space $C^n_{\gamma;\psi}[a,b]$ of continuous f on $(a,b]$ is defined by

$$C^{n}_{\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{R}; f(x) \in C^{n-1}[a,b]; f^{(n)}(x) \in C_{\gamma;\psi}[a,b]\}, \tag{2.4}$$

$0 \leq \gamma < 1$ with the norm

$$||f||_{C^{n}_{\gamma;\psi}[a,b]} = \sum_{k=0}^{n-1} \left|\left|f^{(k)}\right|\right|_{C[a,b]} + \left|\left|f^{(n)}\right|\right|_{C_{\gamma;\psi}[a,b]}. \tag{2.5}$$

Definition 1. [29] Let (a,b) $(-\infty \leq a < b \leq \infty)$ be a finite interval (or infinite) of the real line $\mathbb{R}$ and let $\alpha > 0$. Also, let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ (we denote first derivative as $\frac{d}{dx}\psi(x) = \psi'(x)$) on (a, b). The left-sided fractional integral of a function f with respect to a function ψ on $[a,b]$ is defined by

$$I^{\alpha;\psi}_{+a} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \psi'(x)(\psi(x) - \psi(s))^{\alpha-1} f(s)\, ds. \tag{2.6}$$

The right–sided fractional integral is defined in an analogous form.

Definition 2. [29] Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, let $I = [a,b]$ be an interval such that $-\infty \leq a < b \leq \infty$ and let $f,\ \psi \in C^{n}[a,b]$ be two functions such that $\psi(x)$ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The left-sided ψ–Hilfer fractional derivative ${}^{H}D^{\alpha,\beta;\psi}_{a+}(.)$ of a function f of order α and type $0 \leq \beta \leq 1$, is defined by

$${}^{H}D^{\alpha,\beta;\psi}_{a+} f(x) = I^{\beta(n-\alpha);\psi}_{a+} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^{n} I^{(1-\beta)(n-\alpha);\psi}_{a+} f(x).$$

The right–sided ψ–Hilfer fractional derivative is defined in an analogous form.

Theorem 1. [29] If $f \in C^{1}[a,b]$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then

$$I^{\alpha;\psi}_{a+}\, {}^{H}D^{\alpha,\beta;\psi}_{a+} f(x) = f(x) - \frac{(\psi(x) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} I^{(1-\beta)(1-\alpha);\psi}_{a+} f(a).$$

Theorem 2. [29] If $f \in C^{1}[a,b]$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, we have

$${}^{H}D^{\alpha,\beta;\psi}_{a+} I^{\alpha;\psi}_{a+} f(x) = f(x).$$

The following definitions are useful in the study of Ulam–Hyers and Ulam–Hyers–Rassias stability results.

Definition 3. [3] For each function y satisfying

$$\left| {}^{H}D^{\alpha,\beta;\psi}_{a+} y(x) - f\left(x, y(x), \int_{a}^{x} K(x,\tau,y(\tau),y(\delta(\tau)))d\tau, \int_{a}^{b} H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right)\right| \leq \theta, \tag{2.7}$$

$x \in [a,b]$, where $\theta \geq 0$, there is a solution y_0 of the NMFIDE and a constant $C > 0$ independent of y and y_0 such that

$$|y(x) - y_0(x)| \leq C\theta, \tag{2.8}$$

for all $x \in [a,b]$, then we say that the NMFIDE has the Ulam-Hyers stability.
If instead of θ, in equation (2.7) and (2.8), we have a nonnegative function σ defined on $[a,b]$, then we say that the NMFIDE has the Ulam–Hyers–Rassias stability.

Definition 4. [3] For each function y satisfying

$$\left| {}^{H}D_{a+}^{\alpha,\beta;\psi} y(x) - f\left(x, y(x), \int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau, \int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right)\right| \leq \theta, \qquad (2.9)$$

$x \in [a,b]$, where $\theta \geq 0$, there is a solution y_0 of the NMFIDE and a constant $C > 0$ independent of y and y_0 such that

$$|y(x) - y_0(x)| \leq C\sigma(x), \qquad (2.10)$$

for all $x \in [a,b]$, for some nonnegative function σ defined on $[a,b]$, then we say that the NMFIDE has the so-called semi–Ulam–Hyers stability.

Definition 5. [21] We say that $d : \mathbb{X} \times \mathbb{X} \to [0,\infty]$ is a generalized metric on $\mathbb{X}$ if:

(1) $d(x,y) = 0$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$, for all $x, y \in \mathbb{X}$.
(2) $d(x,z) \leq d(x,y) + d(y,z)$, for all $x, y, z \in \mathbb{X}$.

Theorem 3. (Banach)[21] Let $(\mathbb{X}, d)$ be a generalized complete metric space and $T : \mathbb{X} \to \mathbb{X}$ a strictly contractive operator with Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in \mathbb{X}$, then the following three propositions hold true:

(1) The sequence $(T^n x)_{n\in\mathbb{N}}$ converges to fixed x^* of T;
(2) x^* is the unique fixed point of T in $\mathbb{X}^* = \{y \in \mathbb{X} : d(T^k x, y) < \infty\}$;
(2) If $y \in \mathbb{X}^*$, then

$$d(y,x^*) \leq \frac{1}{1-L} d(Ty,y). \qquad (2.11)$$

3 Ulam–Hyers–Rassias stability

In this section, we present sufficient conditions to obtain the Ulam–Hyers–Rassias stability of the NMFIDE (1.1), where $x \in [a,b]$, for some fixed real numbers a and b.
Consider the space of continuously differentiable functions $C^1[a,b]$ on $[a,b]$ endowed with a Bielecki type metric

$$d(u,v) = \sup_{x\in[a,b]} \frac{|u(x) - v(x))|}{\sigma(x)}, \qquad (3.1)$$

where σ is a non decreasing continuous function $\sigma : [a,b] \to (0,\infty)$. We recall that $(C^1[a,b], d)$ is a complete metric space.

Theorem 4. Let $\delta : [a,b] \to [a,b]$ be a continuous delay function with $\sigma(t) \leq t$, for all $t \in [a,b]$ and $\sigma : [a,b] \to (0,\infty)$ a non decreasing continuous function. In addition, suppose that there is $\xi \in [0,1)$, such that

$$\frac{1}{\Gamma(\alpha)} \int_a^x \psi'(\tau)(\psi(x) - \psi(\tau))^{\alpha-1} \sigma(\tau)\, d\tau \leq \xi\sigma(x), \qquad (3.2)$$

for $x \in [a,b]$, a and b are fixed real numbers. Moreover, suppose that $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$|f(x,u,g_1,g_2) - f(x,v,h_1,h_2)| \leq M(|u-v| + |g_1 - h_1| + |g_2 - h_2|), \qquad (3.3)$$

with $M > 0$, $K : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $H : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are continuous kernels satisfying the Lipschitz condition

$$|K(x,u,g_1,g)-K(x,v,h_1,h)| \leq L_1\,|g-h| \tag{3.4}$$

$$|H(x,u,g_2,g)-H(x,v,h_2,h)| \leq L_2\,|g-h|, \tag{3.5}$$

with $L_1,\ L_2>0$. If $y\in C^1[a,b]$ is such that

$$\left|{}^{H}D_{a+}^{\alpha,\beta;\psi}y(x)-f\left(x,y(x),\int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau,\int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right)\right|\leq\sigma(x), \tag{3.6}$$

$x\in[a,b]$, and $M(\xi+L_1\xi^2+L_2\xi^2)<1$, then there is a unique function, $y_0\in C^1[a,b]$ such that

$${}^{H}D_{+a}^{\alpha,\beta;\psi}y_0(x)=f\left(x,y_0(x),\int_a^x K(x,\tau,y_0(\tau),y_0(\delta(\tau)))d\tau,\int_a^b H(x,\tau,y_0(\tau),y_0(\delta(\tau)))d\tau\right),$$

and

$$|y(x)-y_0(x)|\leq\frac{\xi\sigma(x)}{1-M(\xi+L_1\xi^2+L_2\xi^2)}, \tag{3.7}$$

for all $x\in[a,b]$. Then the NMFIDE (1.1) has the Ulam-Hyers-Rassias stability.

Proof. Applying the fractional integral operator $I_{a+}^{\alpha;\psi}(.)$ on both sides of the NMFIDEs (1.1) and using Theorem (1), we can write

$$\begin{aligned}y(x)=&\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c\\&+I_{a+}^{\alpha;\psi}f\left(x,y(x),\int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau,\int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right).\end{aligned} \tag{3.8}$$

On the other hand, if y satisfies Eq. (3.8), then y satisfies Eq.(1.1). However, applying the fractional derivative ${}^{H}D_{a+}^{\alpha,\beta;\psi}(.)$ on both sides of Eq. (3.8), we have

$$\begin{aligned}{}^{H}D_{a+}^{\alpha,\beta;\psi}y(x)=&{}^{H}D_{a+}^{\alpha,\beta;\psi}\left[\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c\right]\\&+{}^{H}D_{a+}^{\alpha,\beta;\psi}I_{a+}^{\alpha;\psi}f\left(x,y(x),\int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau,\int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right).\end{aligned} \tag{3.9}$$

Using Theorem (2) and the expression

$${}^{H}D_{a+}^{\alpha,\beta;\psi}(\psi(x)-\psi(a))^{\gamma-1}=0,\ 0<\gamma<1,$$

we conclude that, $y(x)$ satisfies initial value problem Eq. (1.1) if, and only if, $y(x)$ satisfies the integral equation

$$\begin{aligned}y(x)=&\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c\\&+I_{a+}^{\alpha;\psi}f\left(x,y(x),\int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau,\int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right).\end{aligned} \tag{3.10}$$

So, consider the operator $T:C^1[a,b]\to C[a,b]$, defined by

$$Tu(x) = \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c + I_{a+}^{\alpha;\psi} f\left(x, u(x), \int_a^x K(x,\tau,u(\tau),u(\delta(\tau)))d\tau, \int_a^b H(x,\tau,u(\tau),u(\delta(\tau)))d\tau\right), \tag{3.11}$$

for all $x \in [a,b]$ and $u \in C^1[a,b]$.
Note that for any continuous function u, Tu is also continuous. In fact,

$$\begin{aligned}
&|Tu(x) - Tu(x_0)| \\
&= \left| \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c + I_{a+}^{\alpha;\psi} f\left(x, u(x), \int_a^x K(x,\tau,u(\tau),u(\delta(\tau)))d\tau, \int_a^b H(x,\tau,u(\tau),u(\delta(\tau)))d\tau\right)\right. \\
&- \frac{(\psi(x_0)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c \\
&\left. - I_{a+}^{\alpha;\psi} f\left(x_0, u(x_0), \int_a^{x_0} K(x_0,\tau,u(\tau),u(\delta(\tau)))d\tau, \int_a^b H(x_0,\tau,u(\tau),u(\delta(\tau)))d\tau\right)\right| \\
&\to 0,
\end{aligned} \tag{3.12}$$

when $x \to x_0$.
We will deduce that the operator T is strictly contractive with respect to the metric Eq. (3.1). Indeed, for all, $u, v \in C^1[a,b]$ we get

$$\begin{aligned}
d(Tu,Tv) &= \sup_{x\in[a,b]} \left| I_{a+}^{\alpha;\psi} f\left(x, u(x), \int_a^x K(x,\tau,u(\tau),u(\delta(\tau)))d\tau, \int_a^b H(x,\tau,u(\tau),u(\delta(\tau)))d\tau\right)\right. \\
&\left. - I_{a+}^{\alpha;\psi} f\left(x_0, u(x_0), \int_a^{x_0} K(x_0,\tau,u(\tau),u(\delta(\tau)))d\tau, \int_a^b H(x_0,\tau,u(\tau),u(\delta(\tau)))d\tau\right)\right| \Big/ \sigma(x) \\
&\le \sup_{x\in[a,b]} \left| I_{a+}^{\alpha;\psi}\left(M\left[|u(x)-v(x)| + \int_a^x \Big| K(x,\tau,u(\tau),u(\delta(\tau))) - K(x,\tau,v(\tau),v(\delta(\tau)))\Big| d\tau \right.\right.\right. \\
&\left.\left.\left. + \int_a^b \Big| H(x,\tau,u(\tau),u(\delta(\tau))) - H(x,\tau,v(\tau),v(\delta(\tau)))\Big| d\tau \right]\right)\right| \Big/ \sigma(x) \\
&\le M \sup_{x\in[a,b]} \frac{I_{a+}^{\alpha;\psi}(|u(x)-v(x)|)}{\sigma(x)} + ML_1 \sup_{x\in[a,b]} \frac{I_{a+}^{\alpha;\psi} \int_a^x |u(\delta(\tau)) - v(\delta(\tau))| d\tau}{\sigma(x)} \\
&+ ML_2 \sup_{x\in[a,b]} \frac{I_{a+}^{\alpha;\psi} \int_a^b |u(\delta(\tau)) - v(\delta(\tau))| d\tau}{\sigma(x)} \\
&\le M\xi \sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} + ML_1 \sup_{x\in[a,b]} \frac{|u(\delta(\tau)) - v(\delta(\tau))|}{\sigma(\tau)} \\
&\sup_{x\in[a,b]} \frac{I_{a+}^{\alpha;\psi}(\int_a^x \sigma(\tau)d\tau)}{\sigma(x)} + ML_2 \sup_{x\in[a,b]} \frac{|u(\delta(\tau)) - v(\delta(\tau))|}{\sigma(\tau)} \\
&\sup_{x\in[a,b]} \frac{I_{a+}^{\alpha;\psi}(\int_a^b \sigma(\tau)d\tau)}{\sigma(x)} \\
&\le M\xi \sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} + ML_1 d(u,v) \sup_{x\in[a,b]} \frac{I_{a+}^{\alpha;\psi}(\xi\sigma(x))}{\sigma(x)}
\end{aligned}$$

$$
\begin{aligned}
&+ML_2 d(u,v)\sup_{x\in[a,b]}\frac{I_{a+}^{\alpha;\psi}(\xi\sigma(x))}{\sigma(x)}\\
&\le M(\xi+L_1\xi^2+L_2\xi^2)d(u,v). \qquad (3.13)
\end{aligned}
$$

As $M(\xi+L_1\xi^2+L_2\xi^2)<1$, it follows that T is strictly contractive. In this way, we can apply the above mentioned Banach fixed-point theorem which allows us to ensure that we have the Ulam-Hyers-Rassias stability for the NMFIDE (1.1).

In fact, from Eq. (3.6), using fractional integral Eq. (2.6) and Theorem 1, we have

$$
\begin{aligned}
&\Bigg|y(x)-\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c\\
&\quad -I_{a+}^{\alpha;\psi}f\left(x,y(x),\int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau,\int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right)\Bigg|\\
&\le I_{a+}^{\alpha;\psi}(\sigma(\tau)),\ x\in[a,b]. \qquad (3.14)
\end{aligned}
$$

Therefore, having in mind Eq. (3.11) and Eq. (3.2), we conclude that

$$
|y(x)-Ty(x)|\le I_{a+}^{\alpha;\psi}(\sigma(\tau))\le\xi\sigma(x),\ x\in[a,b]. \qquad (3.15)
$$

Note that,

$$
\begin{aligned}
-M(\xi+L_1\xi^2+L_2\xi^2)\le -L_1-L_2 &\implies 1-M(\xi+L_1\xi^2+L_2\xi^2)\le 1-L_1-L_2\\
&\implies \frac{1}{1-L_1-L_2}\le\frac{1}{1-M(\xi+L_1\xi^2+L_2\xi^2)}.
\end{aligned}
$$

Consequently, Eq. (3.7) follows directly from the definition of the metric d, Eq. (2.11) and Eq. (3.15).

4 Semi–Ulam–Hyers–Rassias and Ulam–Hyers stabilities

In this section, we state and prove some results related to semi–Ulam–Hyers–Rassias and Ulam–Hyers stabilities for NMFIDE (1.1).

Theorem 5. Let $\delta:[a,b]\to[a,b]$ be a continuous delay function with $\sigma(t)\le t$, for all $t\in[a,b]$ and $\sigma:[a,b]\to(0,\infty)$ a non decreasing continuous function. In addition, suppose that there is $\xi\in[0,1)$, such that

$$
\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(\tau)(\psi(x)-\psi(\tau))^{\alpha-1}\sigma(\tau)\,d\tau\le\xi\sigma(x), \qquad (4.1)
$$

for all $x\in[a,b]$, Moreover, suppose that $f:[a,b]\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$
|f(x,u,g_1,g_2)-f(x,v,h_1,h_2)|\le M(|u-v|+|g_1-h_1|+|g_2-h_2|), \qquad (4.2)
$$

with $M>0$ and the kernel $K:[a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ and $H:[a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ are continuous functions satisfying the Lipschitz condition

$$
|K(x,t,u,w)-K(x,t,v,z)|\le L_1\,|w-z| \qquad (4.3)
$$

$$|H(x,t,u,w)-H(x,t,v,z)| \le L_2 |w-z|, \tag{4.4}$$

with $L_1,\ L_2 > 0$. If $y \in C^1[a,b]$ is such that

$$\left| {}^H D_{a+}^{\alpha,\beta;\psi} y(x) - f\left(x, y(x), \int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau, \int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right)\right| \le \theta, \tag{4.5}$$

$x \in [a,b]$, where $\theta > 0$ and $M(\xi + L_1\xi^2 + L_2\xi^2) < 1$, then there is a unique function, $y_0 \in C^1[a,b]$ such that
${}^H D_{+a}^{\alpha,\beta;\psi} y_0(x) = f\left(x, y_0(x), \int_a^x K(x,\tau,y_0(\tau),y_0(\delta(\tau)))d\tau, \int_a^b H(x,\tau,y_0(\tau),y_0(\delta(\tau)))d\tau\right)$,
and

$$|y(x)-y_0(x)| \le \frac{(b-a)\theta\sigma(x)}{1-M(\xi+L_1\xi^2+L_2\xi^2)\sigma(a)}, \tag{4.6}$$

for all $x \in [a,b]$. Then the NMFIDE (1.1) has the semi–Ulam-Hyers-Rassias stability.

Proof. The first part of the proof follows the same steps as in the proof of Theorem 4. The main purpose here is to choose a general σ and do as previously done in the metric function d, using the same idea as in Eq. (3.10)–Eq. (3.12), and just take it as a constant in all the remaining places.

Consider the operator $T : C^1[a,b] \to C^1[a,b]$, defined by

$$\begin{aligned} Tu(x) = {} & \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c \\ & + I_{a+}^{\alpha;\psi} f\left(x, u(x), \int_a^x K(x,\tau,u(\tau),u(\delta(\tau)))d\tau, \int_a^b H(x,\tau,u(\tau),u(\delta(\tau)))d\tau\right), \end{aligned} \tag{4.7}$$

for all $x \in [a,b]$, $u \in C^1[a,b]$ and by using the same reasoning an in Eq. (3.11)–Eq. (3.12) we conclude that T is strictly contractive with respect to the metric Eq. (3.1), due to the fact that $M(\xi + L_1\xi^2 + L_2\xi^2) < 1$. Thus, we can again apply the Banach fixed-point theorem, which guarantees us that

$$d(y,y_0) \le \frac{1}{1-M(\xi+L_1\xi^2+L_2\xi^2)} d(Ty,y). \tag{4.8}$$

Now, by Eq. (4.5), using fractional integral Eq. (2.6) and Theorem 1, we get

$$\begin{aligned} & \left| y(x) - \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c \right. \\ & \left. \quad - I_{a+}^{\alpha;\psi} f\left(x, y(x), \int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau, \int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right)\right| \\ & \quad \le I_{a+}^{\alpha;\psi}\theta. \end{aligned} \tag{4.9}$$

Therefore, having in mind Eq. (3.11) and Eq. (3.2), we conclude that

$$|y(x)-Ty(x)| \le \theta I_{a+}^{\alpha;\psi}(1) \le \theta\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}. \tag{4.10}$$

Using the Eq. (4.8), and having in mind Eq. (2.6) and that σ is a positive non-decreasing function, it follows

$$\sup_{x\in[a,b]} \frac{|y(x)-y_0(x))|}{\sigma(x)} \le \frac{1}{1-M(\xi+L_1\xi^2+L_2\xi^2)} \sup_{x\in[a,b]} \frac{d(Ty,y)}{\sigma(x)}$$

$$\leq \frac{\theta(\psi(b)-\psi(a))^{\alpha}}{(1-M(\xi+L_1\xi^2+L_2\xi^2))\Gamma(\gamma+1)\sigma(a)}. \tag{4.11}$$

Consequently, Eq. (4.6) follows directly from Eq. (4.11) and this led us to the semi–Ulam–Hyers–Rassias stability of the fractional integro-differential equation under study.

Still having in mind that σ is a positive non-decreasing function, and considering an obvious upper bound in Eq. (4.6), we directly obtain from the last result the following Ulam-Hyers stability of the NMFIDE (1.1).

Corollary 1. Let $\delta : [a,b] \to [a,b]$ be a continuous delay function with $\sigma(t) \leq t$, for all $t \in [a,b]$ and $\sigma : [a,b] \to (0,\infty)$ a non decreasing continuous function. In addition, suppose that there is $\xi \in [0,1)$, such that

$$\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(\tau)(\psi(x)-\psi(\tau))^{\alpha-1}\sigma(s)\,d\tau \leq \xi\sigma(x), \tag{4.12}$$

for $x \in [a,b]$, Moreover, suppose that $f : [a,b]\times\mathbb{C}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$|f(x,u,g_1,g_2)-f(x,v,h_1,h_2)| \leq M(|u-v|+|g_1-h_1|+|g_2-h_2|), \tag{4.13}$$

with $M > 0$ and $K : [a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ and $H : [a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ are continuous kernels satisfying the Lipschitz condition

$$|K(x,t,u,w)-K(x,t,v,z)| \leq L_1\,|w-z| \tag{4.14}$$

$$|H(x,t,u,w)-H(x,t,v,z)| \leq L_2\,|w-z|\,, \tag{4.15}$$

with $L_1,\ L_2 > 0$. If $y \in C^1[a,b]$ is such that

$$\left|{}^{H}D_{a+}^{\alpha,\beta;\psi}y(x)-f\left(x,y(x),\int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau,\int_a^b H(x,\tau,y(\tau),y(\delta(\tau)))d\tau\right)\right| \leq \theta, \tag{4.16}$$

$x \in [a,b]$, where $\theta > 0$ and $M(\xi+L_1\xi^2+L_2\xi^2) < 1$, then there is a unique function, $y_0 \in C^1[a,b]$ such that
${}^{H}D_{+a}^{\alpha,\beta;\psi}y_0(x) = f\left(x,y_0(x),\int_a^x K(x,\tau,y_0(\tau),y_0(\delta(\tau)))d\tau,\int_a^b H(x,\tau,y_0(\tau),y_0(\delta(\tau)))d\tau\right)$,
and

$$|y(x)-y_0(x)| \leq \frac{\theta(\psi(b)-\psi(a))^{\alpha}\sigma(b)}{(1-M(\xi+L_1\xi^2+L_2\xi^2))\Gamma(\gamma+1)\sigma(a)}, \tag{4.17}$$

$x \in [a,b]$.
This means that under above conditions, the NMFIDE (1.1) has the Ulam-Hyers stability.

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