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Abstract. The aim of this study is to investigate the existence and other properties of solution of nonlinear fractional integro–differential equations with constant coefficient. Also with the help of Pachpatte's inequality, we prove the continuous dependence of the solutions.

Mathematics Subject Classification. Primary 26A33; Secondary 34G20, 34A08, 34A12.

Keywords. Fractional integro–differential equation, existence of solution, continuous dependence, fixed point theorem, Pachpatte's inequality.

1. Introduction

Fractional differential equations have been recently used as effective tools in the modeling of many phenomena in various fields of applied sciences and engineering such as acoustic control, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, proteins, electromagnetics, optics, medicine, economics, astrophysics, chemical engineering, chaotic dynamics, statistical physics and so on (see [1,3,5,6,8,12,18,19,27,30] and the references therein). Many problems can be modeled by fractional integro–differential equations from various sciences and engineering applications.

Recently, many researchers have studied the Cauchy problem and long-time behavior for nonlinear fractional differential and integro–differential equations and obtained many interesting results using all kinds of fixed point theorems, for example, by Aghajani et al. [2], Balachandrann and Park [4], Cabrera et al. [7], Dong et al. [10], Furati and Tatar [11], Jagtap and Kharat [13], Kharat [14], Kendre et al. [15–17], Liang et al. [20], N'Guérékata [22,23], Pierri and O'Regan [26], Ren et al. [28], Wang and Li [33], Zhou et al. [34,35], Zhou and Jiao [36] and the references therein.

Fractional differential equations with constant coefficients are used to describe many physical and chemical problems [27] such as the motion of a large thin plate in a Newtonian fluid, the process of cooling a semi-infinite

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body by radiation, the phenomena in electromagnetic, acoustic, viscoelasticity, electrochemistry and material science and so on. Therefore, it is worth to study a nonlinear fractional differential equations with constant coefficient.

In [31], using generalized Banach fixed point theorem, Tate et al. discussed the existence and interval of existence of solutions, uniqueness, continuous dependence of solutions on initial conditions, estimates on solutions and continuous dependence on parameters and functions involved in the nonlinear fractional differential equation with constant coefficient $\lambda > 0$ of the type:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = \lambda x(t) + f(t, x(t)), \ t \in [0, T], \ T > 0, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\alpha}(0 < \alpha < 1)$ denotes the caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [32], using Banach fixed point theorem coupled with Bielecki-type norm and the integral inequality, Tidke investigated the existence, uniqueness and other properties of solutions of fractional semilinear evolution equation of the type:

$$\begin{cases} D^{q}x(t) = A(t)x(t) + f(t, x(t)), \ t \in J = [0, b], \\ x(0) = x_0, \end{cases}$$

where $0 < q < 1$, D^q denotes the Caputo fractional derivative of order q, the unknown $x(.)$ takes the values in the Banach space X; $f \in C(J \times X, X)$, and A(t) is a bounded linear operator on X and x_0 is a given element of X.

Motivated by the above-mentioned work, in this paper, we investigate the existence and interval of existence, uniqueness, continuous dependence of solutions on initial conditions of a nonlinear fractional integro–differential equations with constant coefficient $\lambda > 0$ of the type:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = \lambda x(t) + f\left(t, x(t), \int_0^t h(t, s)x(s)\mathrm{d}s\right), \ t \in J = [0, T], \ T > 0, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.2}$$

where ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order $0 < \alpha < 1$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : J \times J \to \mathbb{R}$ are given continuous functions.

We further derive an estimate on solutions and continuous dependence on parameters and functions involved in the right hand side of Eq. (1.2). Finally, one illustrative example is given to demonstrate the theoretical results.

2. Preliminaries

Here, we present some definitions, notations and results from [8,21,27,29] which are used throughout this paper.

Let $C(J, \mathbb{R})$ be the Banach space of continuous functions from J into $\mathbb{R}$ with the supremum norm $||.||$.

Definition 2.1. A real-valued function $f(t)(t > 0)$ is said to be in space $C_\mu(\mu \in \mathbb{R})$, if there exists a real number $p > \mu$ such that $f(t) = t^p g(t)$, where $g \in C[0, \infty)$.

Definition 2.2. A real-valued function $f(t)(t > 0)$ is said to be in the space $C_\mu^n, n \in \mathbb{N} \cup \{0\}$, if $f^n \in C_\mu$.

Definition 2.3. Let $f \in C_\mu(\mu \geq -1)$, then the (left-sided) Riemann–Liouville fractional integral of order $\alpha > 0$ of the function f is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,\ t > 0 \ and\ I^0 f(t) = f(t),$$

where $\Gamma(.)$ is Euler's Gamma function.

Definition 2.4. The (left-sided) Caputo fractional derivative of order $\alpha > 0$ of the function $f \in C_{-1}^n$ $(n \in \mathbb{N} \cup \{0\})$, is given by:

$${}^cD^\alpha f(t) = \begin{cases} f^n(t), & \text{if } \alpha = n \\ I^{n-\alpha} f^n(t), & \text{if } n-1 < \alpha < n,\ n \in \mathbb{N}, \end{cases}$$

where $n = [\alpha] + 1$ denotes the integer part of the real number α.

Note that $I^\alpha\ {}^cD^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0^+)}{k!} t^k,\ n-1 < \alpha \leq n,\ n \in \mathbb{N}$.

Definition 2.5. The (left-sided) Riemann–Liouville fractional derivative of order $\alpha > 0$ of the function $f \in C_{-1}^n$ $(n \in \mathbb{N} \cup \{0\})$, is given by:

$$D^\alpha f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t),\ n-1 < \alpha \leq n,\ n \in \mathbb{N}.$$

Following relation holds between Caluto and Riemann–Liouville fractional derivatives:

$${}^cD^\alpha f(t) = D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{f^k(0^+)}{k!} t^k \right),\ n-1 < \alpha \leq n,\ n \in \mathbb{N}.$$

Definition 2.6. The function $\mathbb{E}_\alpha(\alpha > 0)$ defined by $\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, is called Mittag–Leffler function of order α.

Lemma 2.7. *Let* $\alpha, \beta \in [0, \infty)$. *Then*

$$\int_0^t s^{\alpha-1}(t-s)^{\beta-1}\,\mathrm{d}s = t^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The following Pachpatte's inequality plays an important role in obtaining our main results.

Theorem 2.8 ([25] p. 39). *Let* $u(t)$, $f(t)$ *and* $q(t)$ *be nonnegative continuous functions defined on* $\mathbb{R}_+$, *and* $n(t)$ *be a positive and nondecreasing continuous function defined on* $\mathbb{R}_+$ *for which the inequality*

$$u(t) \leq n(t) + \int_0^t f(s) \left[u(s) + \int_0^s q(\tau) u(\tau)\,\mathrm{d}\tau \right] \mathrm{d}s,$$

holds for $t \in \mathbb{R}_+$. *Then*

$$u(t) \leq n(t)\left[1 + \int_0^t f(s) exp\left(\int_0^s [f(\tau) + q(\tau)]\, \mathrm{d}\tau\right) \mathrm{d}s\right]$$

for $t \in \mathbb{R}_+$.

We will use the generalized Banach fixed point theorem to prove the existence results.

Theorem 2.9 [9]. *Let* U *be a nonempty closed subset of a Banach space* E, *and let* $\alpha_n \geq 0$, $n \in \mathbb{N} \cup \{0\}$, *be a sequence such that* $\sum_{n=0}^{\infty} \alpha_n$ *converges. Moreover, let the mapping* $A : U \to U$ *satisfy the inequality*

$$\|A^n u - A^n v\| \leq \alpha_n \|u - v\|$$

for every $n \in \mathbb{N} \cup \{0\}$, *and every* $u, v \in U$. *Then,* A *has a uniquely defined fixed point* u^*. *Furthermore, the sequence* $\{A^n u_0\}_{n=1}^{\infty}$ *converges to this fixed point* u^* *for every* $u_0 \in U$.

3. Existence Results and Interval of Existence

The following lemma deals with the equivalence of a nonlinear fractional integro–differential equation (1.2).

Lemma 3.1. *If the function* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function, then a nonlinear fractional integro–differential equation* (1.2) *is equivalent to the integral equation,*

$$\begin{aligned} x(t) = x_0 &+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, \mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), \int_0^s h(t,\tau) x(\tau) \mathrm{d}\tau\right) \mathrm{d}s, \\ & t \in J. \end{aligned} \tag{3.1}$$

Proof. Let $x(t)$ be a solution of Eq. (1.2). Define

$$z(t) = \lambda x(t) + f\left(t, x(t), \int_0^t h(t,s) x(s) \mathrm{d}s\right).$$

Then

$$z(t) = {}^c D^{\alpha} x(t).$$

Since

$${}^c D^{\alpha} x(t) = D^{\alpha}(x(t) - x_0),$$

where D^{α} is Riemann–Liounille fractional derivative of order α with lower limit 0, then we get

$$z(t) = D^{\alpha}(x(t) - x_0) = \frac{d}{dt} I^{1-\alpha}(x(t) - x_0).$$

This gives

$$I^1 z(t) = I^{1-\alpha}(x(t) - x_0) + k,$$

where k is any constant. since $z(t)$ and $x(t)-x_0$ are both continuous functions, for $t=0$ we get $k=0$. This gives

$$I^1z(t)=I^{1-\alpha}(x(t)-x_0).$$

Operating Riemann–Liouville fractional differential operator $D^{1-\alpha}$ on both sides, we obtain

$$\begin{aligned}x(t)-x_0&=D^{1-\alpha}I^1z(t)\\&=D^1I^{\alpha}I^1z(t)\\&=D^1I^{1+\alpha}z(t)\\&=I^{\alpha}z(t)\end{aligned}$$

Using the definition of $z(t)$, we obtain (3.1).

Conversely, suppose that $x(t)$ is the solution of the Eq. (3.1). Then, it can be written as

$$x(t)=x_0+I^{\alpha}z(t), \tag{3.2}$$

where $z(t)=\lambda x(t)+f\left(t,x(t),\int_0^t h(t,s)x(s)\mathrm{d}s\right)$. Since $z(t)$ is continuous and x_0 is constant, operating the Caputo fractional differential operator ${}^cD^{\alpha}$ on both sides of Eq. (3.2) we obtain

$${}^cD^{\alpha}x(t)={}^c D^{\alpha}x_0+{}^c D^{\alpha}I^{\alpha}z(t)=z(t).$$

This gives

$${}^cD^{\alpha}x(t)=\lambda x(t)+f\left(t,x(t),\int_0^t h(t,s)x(s)\mathrm{d}s\right).$$

From (3.2), we get $x(0)=x_0$. This proves that $x(t)$ is the solution of Eq. (1.2). □

Theorem 3.2. *Let $T>0$ and let $\xi>0$ be a constant such that $0\in[x_0-\xi,x_0+\xi]$. Assume that $f:J\times[x_0-\xi,x_0+\xi]\times[x_0-\xi,x_0+\xi]\to\mathbb{R}$ satisfies the following condition:*

(H1) There exists a constant $L>0$ such that

$$|f(t,x,y)-f(t,\bar{x},\bar{y})|\leq L(|x-\bar{x}|+|y-\bar{y}|)$$

Let

$$\chi=min\left\{T,\left[\frac{\Gamma(\alpha+1)\xi}{(\xi+|x_0|)(\lambda+L(1+Th_T))+M}\right]^{\frac{1}{\alpha}}\right\}$$

where $h_T=Sup\{|h(t,s)|\,|0<s<t<T\}$, $M=Sup_{t\in J}|f(t,0,0)|$. Then, the equation (1.2) has a unique solution $x:[0,\chi]\to\mathbb{R}$.

Proof. Define the set $U=\{x\in C([0,\chi],\mathbb{R}):x(0)=x_0,||x-x_0||\leq\xi\}$. Since $x_0\in U$, U is nonempty. Also, U is a closed, bounded and convex subset of Banach space $C([0,\chi],\mathbb{R})$. On U, we define an operator A by

$$\begin{aligned}Ax(t)&=x_0+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)\,\mathrm{d}s\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,x(s),\int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right)\,\mathrm{d}s,\ t\in[0,\chi].\end{aligned}$$

Now, we prove that A maps the set U to itself. Let us take any $x \in U$ and $t \in [0, \chi]$. Then,

$$\begin{aligned}|Ax(t) - x_0| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s)| \, \mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right)\right| \mathrm{d}s \\ &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s)| \, \mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigg| f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right) \\ &\quad - f(s,0,0)\bigg| \, \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,0,0)| \, \mathrm{d}s\end{aligned}$$

Using (H1) and Definition of U, for any $t \in [0, \chi]$,

$$|x(t)| \leq |x(t) - x_0| + |x_0| \leq \xi + |x_0| \tag{3.3}$$

$$\begin{aligned}&\left| f\left(s, x(s), \int_0^s h(t,\tau)x(\tau)\mathrm{d}\tau\right) - f(s,0,0)\right| \\ &\quad \leq L\left[|x(t)| + \int_0^s |h(s,\tau)| \, |x(\tau)| \, \mathrm{d}\tau\right] \\ &\quad \leq L(\xi + |x_0|)(1 + Th_T)\end{aligned} \tag{3.4}$$

Therefore

$$\begin{aligned}|Ax(t) - x_0| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\xi + |x_0|) \, \mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L(1 + Th_T)(\xi + |x_0|) \, \mathrm{d}s \\ &\quad + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, \mathrm{d}s \\ &\leq \left\{\frac{(\lambda + L(1 + Th_T))(\xi + |x_0|) + M}{\Gamma(\alpha+1)}\right\} t^\alpha \\ &\leq \left\{\frac{(\lambda + L(1 + Th_T))(\xi + |x_0|) + M}{\Gamma(\alpha+1)}\right\} \chi^\alpha \\ &\leq \left\{\frac{(\lambda + L(1 + Th_T))(\xi + |x_0|) + M}{\Gamma(\alpha+1)}\right\} \\ &\quad \times \left\{\frac{\Gamma(\alpha+1)\xi}{(\xi + |x_0|)(\lambda + L(1 + Th_T)) + M}\right\} \\ &\leq \xi.\end{aligned}$$

we note that, for $0 \leq t_1 \leq t_2 \leq \chi$,

$$
\begin{aligned}
&|Ax(t_1) - Ax(t_2)| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\Bigg|\lambda \int_0^{t_1} (t_1 - s)^{\alpha-1} x(s)\, \mathrm{d}s \\
&\qquad + \int_0^{t_1} (t_1 - s)^{\alpha-1} f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right) \mathrm{d}s \\
&\qquad -\lambda \int_0^{t_2} (t_2 - s)^{\alpha-1} x(s)\, \mathrm{d}s \\
&\qquad - \int_0^{t_2} (t_2 - s)^{\alpha-1} f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right) \mathrm{d}s\Bigg| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\Bigg|\lambda \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\} x(s)\, \mathrm{d}s \\
&\qquad + \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\} f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right) \mathrm{d}s \\
&\qquad -\lambda \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} x(s)\, \mathrm{d}s \\
&\qquad - \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right) \mathrm{d}s\Bigg| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\lambda \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\}(|x(s) - x_0| + |x_0|)\, \mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\} \\
&\qquad \times \left\{\left|f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right) - f(s,0,0)\right| + |f(s,0,0)|\right\} \mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\alpha)}\lambda \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}(|x(s) - x_0| + |x_0|)\, \mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left\{\left|f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right) - f(s,0,0)\right|\right. \\
&\qquad \left. + |f(s,0,0)|\right\} \mathrm{d}s
\end{aligned}
$$

Using (3.3) and (3.4), we get

$$
\begin{aligned}
&|Ax(t_1) - Ax(t_2)| \\
&\quad \leq \left\{\frac{(\lambda + L(1 + Th_T))(\xi + x_0) + M}{\Gamma(\alpha)}\right\} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\}\, \mathrm{d}s \\
&\qquad + \left\{\frac{(\lambda + L(1 + Th_T))(\xi + x_0) + M}{\Gamma(\alpha)}\right\} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\, \mathrm{d}s \\
&\quad \leq \left\{\frac{(\lambda + L(1 + Th_T))(\xi + x_0) + M}{\Gamma(\alpha + 1)}\right\}\{2(t_2 - t_1)^{\alpha} + t_1^{\alpha} - t_2^{\alpha}\}.
\end{aligned}
$$

This shows that Ax is continuous.

Thus for any $x \in U$, we have $Ax \in C([0,\chi],\mathbb{R})$, $Ax(x_0) = x_0$ and $||Ax - x_0|| \leq \xi$. This proves that $Ax \in U$ whenever $x \in U$, i.e., A maps the set U into itself.

The next step is to prove that, for every $n \in \mathbb{N} \cup \{0\}$, and every $x, y \in U$, we have

$$\|A^n x - A^n y\| \leq \frac{[(\lambda + L(1+Th_T))t^\alpha]^n}{\Gamma(n\alpha+1)} \|x-y\|\,,\ t \in [0,\chi]. \tag{3.5}$$

This can be seen by induction. For $n = 0$, the inequality (3.5) is trivially true. We assume that (3.5) is true for $n = m-1$ and prove it for $n = m$. Using definition of operator A and hypothesis (H1), we have

$$\begin{aligned}
&|A^m x(t) - A^m y(t)| \\
&\quad = \left|A(A^{m-1}x(t)) - A(A^{m-1}y(t))\right| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\bigg|\lambda \int_0^t (t-s)^{\alpha-1} A^{m-1}x(s)\,\mathrm{d}s \\
&\qquad + \int_0^t (t-s)^{\alpha-1} f\left(s, A^{m-1}x(s), \int_0^s h(s,\tau)A^{m-1}x(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\qquad - \lambda \int_0^t (t-s)^{\alpha-1} A^{m-1}y(s)\,\mathrm{d}s \\
&\qquad - \int_0^t (t-s)^{\alpha-1} f\left(s, A^{m-1}y(s), \int_0^s h(s,\tau)A^{m-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s\bigg| \\
&|A^m x(t) - A^m y(t)| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\lambda \int_0^t (t-s)^{\alpha-1}\left|A^{m-1}x(s) - A^{m-1}y(s)\right|\mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigg| f\left(s, A^{m-1}x(s), \int_0^s h(s,\tau)A^{m-1}x(\tau)\mathrm{d}\tau\right) \\
&\qquad - f\left(s, A^{m-1}y(s), \int_0^s h(s,\tau)A^{m-1}y(\tau)\mathrm{d}\tau\right)\bigg|\mathrm{d}s
\end{aligned} \tag{3.6}$$

Using hypothesis (H1) and (3.5) in (3.6) for $n = m-1$, we get

$$\begin{aligned}
&|A^m x(t) - A^m y(t)| \\
&\quad \leq \frac{\lambda}{\Gamma(\alpha)} \frac{(\lambda + L(1+Th_T))^{m-1}}{\Gamma((m-1)\alpha+1)} \|x-y\| \int_0^t (t-s)^{\alpha-1} s^{\alpha m-\alpha}\,\mathrm{d}s \\
&\qquad + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigg\{\left|A^{m-1}x(s) - A^{m-1}y(s)\right| \\
&\qquad + \int_0^s |h(s,\tau)|\left|A^{m-1}x(s) - A^{m-1}y(s)\right|\mathrm{d}\tau\bigg\}\mathrm{d}s \\
&\quad \leq \frac{(\lambda + L(1+Th_T))}{\Gamma(\alpha)} \frac{(\lambda + L(1+Th_T))^{m-1}}{\Gamma((m-1)\alpha+1)} \|x-y\| \int_0^t (t-s)^{\alpha-1} s^{\alpha m-\alpha}\,\mathrm{d}s \\
&\quad \leq \frac{[(\lambda + L(1+Th_T))t^\alpha]^m}{\Gamma(m\alpha+1)} \|x-y\|.
\end{aligned}$$

which is our desired inequality (3.5). Hence, we have

$$\|A^n x - A^n y\| \le \frac{[(\lambda + L(1+Th_T))\chi^\alpha]^n}{\Gamma(n\alpha+1)} \|x-y\|, \ \textit{for all } n \in \mathbb{N}\cup\{0\}.$$

By definition 2.6, we have

$$\sum_{n=0}^{\infty} \frac{[(\lambda + L(1+Th_T))\chi^\alpha]^n}{\Gamma(n\alpha+1)} = \mathbb{E}_\alpha((\lambda + L(1+Th_T))\chi^\alpha).$$

We have proved that the operator A satisfies all the conditions of Theorem 2.9 and hence, A has a unique fixed point $x : [0,\chi] \to \mathbb{R}$ which is the solution of Eq.(1.2). □

Remark 3.3. Hereafter, to study the other properties of solutions of Eq.(1.2), we take $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $C(J,\mathbb{R})$- the Banach space endowed with the supremum norm $||.||$.

4. Estimates on the Solutions

The following theorem contains the estimate on the solution of Eq. (1.2).

Theorem 4.1. *Suppose that the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the hypothesis (H1). If $x(t), t \in J$ is any solution of Eq. (1.2) then*

$$|x(t)| \le \left(|x_0| + \frac{MT^\alpha}{\Gamma(\alpha+1)}\right)\left[1 + \frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)} exp\left\{\frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)} + \frac{Lh_T T}{(\lambda+L)}\right\}\right],$$

$t \in J$. where $M = Sup_{t\in J}|f(t,0,0)|$ and $h_T = Sup\{|h(t,s)| : 0 < s < t < T\}$.

Proof. Let $x(t)$ be any solution of (1.2), then

$$\begin{aligned} {}^cD^\alpha x(t) &= \lambda x(t) + f\left(t, x(t), \int_0^t h(t,s)x(s)\mathrm{d}s\right), \ x(0) = x_0 \\ \Rightarrow x(t) &= x_0 + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, x(s), \int_0^s h(t,\tau)x(\tau)\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned}$$

Therefore,

$$\begin{aligned} |x(t)| \le |x_0| &+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)|\,\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, x(s), \int_0^s h(s,\tau)x(\tau)\mathrm{d}\tau\right) - f(s,0,0)\right|\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\,\mathrm{d}s \end{aligned}$$

Using hypothesis (H1), for any $t \in J$, we get

$$|x(t)| \le |x_0| + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)|\,\mathrm{d}s$$

$$
\begin{aligned}
&+\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|x(s)|\,\mathrm{d}s\\
&+\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s |h(s,\tau)|\,|x(s)|\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&+\frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s\\
\le{}& |x_0| + \frac{M}{\Gamma(\alpha+1)}t^\alpha + \frac{(\lambda+L)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|x(s)|\,\mathrm{d}s\\
&+\frac{Lh_T}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s |x(s)|\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
\le{}& |x_0| + \frac{M}{\Gamma(\alpha+1)}t^\alpha + \int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1}\Big[|x(s)|\\
&+\int_0^s \frac{Lh_T}{(\lambda+L)}\,|x(\tau)|\,\mathrm{d}\tau\Big]\,\mathrm{d}s
\end{aligned}
\tag{4.1}
$$

Applying Pachpattes inequality given in the Theorem 2.8 to the equation (4.1) with

$$
u(t)=x(t),\ n(t)=|x_0|+\frac{M}{\Gamma(\alpha+1)}t^\alpha,\ f(s)=\frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1},
$$

$$
q(\tau)=\frac{Lh_T}{(\lambda+L)},
$$

we obtain

$$
\begin{aligned}
|x(t)| \le{}& \Big(|x_0|+\frac{Mt^\alpha}{\Gamma(\alpha+1)}\Big)\\
&\times\Big[1+\int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1} exp\Big(\int_0^s \Big\{\frac{(\lambda+L)}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}\\
&+\frac{Lh_T}{(\lambda+L)}\Big\}\mathrm{d}\tau\Big)\mathrm{d}s\Big]\\
\le{}& \Big(|x_0|+\frac{MT^\alpha}{\Gamma(\alpha+1)}\Big)\Big[1+\frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)} exp\Big\{\frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)}+\frac{Lh_T T}{(\lambda+L)}\Big\}\Big].
\end{aligned}
$$

This gives

$$
|x(t)| \le \Big(|x_0|+\frac{MT^\alpha}{\Gamma(\alpha+1)}\Big)\Big[1+\frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)} exp\Big\{\frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)}+\frac{Lh_T T}{(\lambda+L)}\Big\}\Big].
$$

□

5. Continuous Dependence and Uniqueness of Solutions

Theorem 5.1. *Suppose that the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the hypothesis (H1). Let $x_1(t)$ and $x_2(t)$ be the solutions of equation*

$$
{}^cD^\alpha x(t) = \lambda x(t) + f\left(t, x(t), \int_0^t h(t,s)x(s)\mathrm{d}s\right),\ t \in J, \tag{5.1}
$$

corresponding to $x(0) = x_0$ and $x(0) = x_0^*$ respectively. Then

$$||x_1 - x_2|| \leq |x_0 - x_0^*| \left[1 + \frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)} exp\left\{\frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)} + \frac{Lh_T T}{(\lambda+L)}\right\}\right] \quad (5.2)$$

Proof. Let $x_1(t)$ and $x_2(t)$ be the solutions of Eq. (5.1) corresponding to $x(0) = x_0$ and $x(0) = x_0^*$, respectively. Therefore,

$$^cD^\alpha x_1(t) = \lambda x_1(t) + f\left(t, x_1(t), \int_0^t h(t,s)x_1(s)\mathrm{d}s\right), \; x_1(0) = x_0$$

and

$$^cD^\alpha x_2(t) = \lambda x_2(t) + f\left(t, x_2(t), \int_0^t h(t,s)x_2(s)\mathrm{d}s\right), \; x_2(0) = x_0^*.$$

This implies

$$x_1(t) = x_0 + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_1(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, x_1(s), \int_0^s h(t,\tau)x_1(\tau)\mathrm{d}\tau\right)\mathrm{d}s,$$

and

$$x_2(t) = x_0^* + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_2(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, x_2(s), \int_0^s h(t,\tau)x_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s.$$

Using the hypothesis (H1), for any $t \in [0,T]$, we obtain

$$\begin{aligned}|x_1(t) - x_2(t)| \leq |x_0 - x_0^*| &+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x_1(s) - x_2(s)|\,\mathrm{d}s \\ &+ \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x_1(s) - x_2(s)|\,\mathrm{d}s \\ &+ \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s |h(s,\tau)|\,|x_1(\tau) - x_2(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s \\ \leq |x_0 - x_0^*| &+ \frac{(\lambda+L)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x_1(s) - x_2(s)|\,\mathrm{d}s \\ &+ \frac{Lh_T}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s |x_1(\tau) - x_2(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s\end{aligned}$$

Therefore,

$$\begin{aligned}|x_1(t) - x_2(t)| \leq |x_0 - x_0^*| + \int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1}\Big[|x_1(s) - x_2(s)| \\ + \int_0^s \frac{Lh_T}{(\lambda+L)}|x_1(\tau) - x_2(\tau)|\,\mathrm{d}\tau\Big]\mathrm{d}s\end{aligned} \quad (5.3)$$

Applying Pachpatte's inequality given in the Theorem 2.8 to the inequation (5.3) with

$$u(t) = |x_1(t) - x_2(t)|\,,\; n(t) = |x_0 - x_0^*|\,,\; f(s) = \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1},$$

$$q(\tau) = \frac{Lh_T}{(\lambda+L)},$$

we obtain

$$\begin{aligned}|x_1(t) - x_2(t)| \le |x_0 - x_0^*| & \\ \times\Bigg[1 + \int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1} exp\Bigg(\int_0^s \Bigg\{\frac{(\lambda+L)}{\Gamma(\alpha)}(s-\tau)^{\alpha-1} \\ +\frac{Lh_T}{(\lambda+L)}\Bigg\}\mathrm{d}\tau\Bigg)\mathrm{d}s\Bigg] \\ \le |x_0 - x_0^*|\left[1 + \frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)} exp\left\{\frac{(\lambda+L)T^\alpha}{\Gamma(\alpha+1)} + \frac{Lh_T T}{(\lambda+L)}\right\}\right].\end{aligned}$$

This gives the inequality (5.2). □

Remark 5.2. The inequality (5.2) shows continuous dependence of solution of equation (1.2) on initial conditions and also it gives the uniqueness. The uniqueness follows by putting $x_0 = x_0^*$ in (5.2).

6. Continuous Dependence on Functions Involved and Parameters in Nonlinear Fractional Integro–Differential Equation

This section deals with the study of continuous dependence of solution of a Eq. (1.2) on functions and parameters involved therein.

Consider the Eq. (1.2) and

$$^cD^\alpha y(t) = \lambda y(t) + F\left(t, y(t), \int_0^t h(t,s)y(s)\mathrm{d}s\right),\; y(0) = y_0 \in \mathbb{R} \qquad (6.1)$$

where $F : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

In the following theorem, we prove continuous dependence of solutions of Eq. (1.2) on the functions involved in right-hand side of Eq. (1.2).

Theorem 6.1. *Suppose that f in (1.2) satisfies the hypothesis (H1). Let $y(t)$ be a solution of Eq. (6.1) and suppose that*

$$|f(t, y(t), \bar{y}(t)) - F(t, y(t), \bar{y}(t))| \le \varepsilon,\; t \in J \text{ and } |x_0 - y_0| < \delta, \qquad (6.2)$$

where $\varepsilon, \delta > 0$ are arbitrary small constants. Then the solution $x(t)$ of the Eq. (1.2) depends continuously on the functions involved therein.

Proof. Let $x(t)$ and $y(t)$ be any solution of (1.2) and (6.1) respectively. Then

$$^cD^\alpha x(t) = \lambda x(t) + f\left(t, x(t), \int_0^t h(t,s)x(s)\mathrm{d}s\right),\; x(0) = x_0$$

and

$$^{c}D^{\alpha}y(t)=\lambda y(t)+F\left(t,y(t),\int_0^t h(t,s)y(s)\mathrm{d}s\right),\ y(0)=y_0$$

This implies

$$x(t)=x_0+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\,\mathrm{d}s \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\left(s,x(s),\int_0^s h(t,\tau)x(\tau)\mathrm{d}\tau\right)\,\mathrm{d}s,$$

and

$$y(t)=y_0+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,\mathrm{d}s \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F\left(s,y(s),\int_0^s h(t,\tau)y(\tau)\mathrm{d}\tau\right)\,\mathrm{d}s.$$

Using hypothesis, we obtain

$$\begin{aligned}|x(t)-y(t)| &\le |x_0-y_0|+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|x(s)-y(s)|\,\mathrm{d}s \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigg|f\left(s,x(s),\int_0^s h(t,\tau)x(\tau)\mathrm{d}\tau\right) \\ &\quad -F\left(s,y(s),\int_0^s h(t,\tau)y(\tau)\mathrm{d}\tau\right)\bigg|\,\mathrm{d}s \\ &\le |x_0-y_0|+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|x(s)-y(s)|\,\mathrm{d}s \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigg|f\left(s,x(s),\int_0^s h(t,\tau)x(\tau)\mathrm{d}\tau\right) \\ &\quad -f\left(s,y(s),\int_0^s h(t,\tau)y(\tau)\mathrm{d}\tau\right)\bigg|\,\mathrm{d}s \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigg|f\left(s,y(s),\int_0^s h(t,\tau)y(\tau)\mathrm{d}\tau\right) \\ &\quad -F\left(s,y(s),\int_0^s h(t,\tau)y(\tau)\mathrm{d}\tau\right)\bigg|\,\mathrm{d}s \\ &\le |x_0-y_0|+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|x(s)-y(s)|\,\mathrm{d}s \\ &\quad +\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|x(s)-y(s)|\,\mathrm{d}s \\ &\quad +\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s |h(s,\tau)|\,|x(\tau)-y(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigg|f(s,y(s),\int_0^s h(t,\tau)y(\tau)\mathrm{d}\tau) \\ &\quad -F(s,y(s),\int_0^s h(t,\tau)y(\tau)\mathrm{d}\tau)\bigg|\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\le \left\{\delta + \frac{\varepsilon}{\Gamma(\alpha+1)}t^{\alpha}\right\} + \frac{(\lambda+L)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|x(s)-y(s)\right| \mathrm{d}s \\
&\quad + \frac{Lh_T}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s |x(\tau)-y(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\le \left\{\delta + \frac{\varepsilon}{\Gamma(\alpha+1)}t^{\alpha}\right\} + \int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1}\Bigg[|x(s)-y(s)| \\
&\quad + \int_0^s \frac{Lh_T}{(\lambda+L)}|x(\tau)-y(\tau)|\,\mathrm{d}\tau\Bigg]\mathrm{d}s
\end{aligned}
$$

Applying Pachpatte's inequality given in the Theorem 2.8 with

$$
u(t) = |x(t)-y(t)|,\; n(t) = \left\{\delta + \frac{\varepsilon}{\Gamma(\alpha+1)}t^{\alpha}\right\},\; f(s) = \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1},
$$

$$
q(\tau) = \frac{Lh_T}{(\lambda+L)},
$$

we obtain

$$
\begin{aligned}
|x(t)-y(t)| &\le \left\{\delta + \frac{\varepsilon}{\Gamma(\alpha+1)}t^{\alpha}\right\} \\
&\quad \times\Bigg[1 + \int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1} exp\Bigg(\int_0^s \Bigg\{\frac{(\lambda+L)}{\Gamma(\alpha)}(s-\tau)^{\alpha-1} \\
&\quad + \frac{Lh_T}{(\lambda+L)}\Bigg\}\mathrm{d}\tau\Bigg)\mathrm{d}s\Bigg] \\
&\le \left\{\delta + \frac{\varepsilon}{\Gamma(\alpha+1)}t^{\alpha}\right\} \\
&\quad \times\left[1 + \int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1} exp\left(\frac{(\lambda+L)s^{\alpha}}{\Gamma(\alpha)} + \frac{Lh_T s}{(\lambda+L)}\right)\mathrm{d}s\right].
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
|x(t)-y(t)| &\le \left\{\delta + \frac{\varepsilon}{\Gamma(\alpha+1)}t^{\alpha}\right\}\Bigg[1 \\
&\quad + \int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(t-s)^{\alpha-1} exp\left(\frac{(\lambda+L)s^{\alpha}}{\Gamma(\alpha)} + \frac{Lh_T s}{(\lambda+L)}\right)\mathrm{d}s\Bigg].
\end{aligned} \tag{6.3}
$$

From the inequality (6.3), it follows that the solution $x(t)$ of Eq. (1.2) depends continuously on the functions involved in the right hand side of equation (1.2). If $\varepsilon = 0$ then the inequality (6.3) gives continuous dependence of solutions on initial conditions and we also note that as $\varepsilon, \delta > 0$ were arbitrary, by taking $\varepsilon, \delta \to 0^+$, we have $x \to y$, where $x : J \to \mathbb{R}$ and $y : J \to \mathbb{R}$ are the solutions of (1.2) and (6.1), respectively. □

Next, we consider the differential equation of fractional order:

$$
{}^cD^{\alpha}x_1(t) = \lambda x_1(t) + H\left(t, x_1(t), \int_0^t h(t,s)x_1(s)\mathrm{d}s, \delta_1\right),\; x_1(0) = x_0 \tag{6.4}
$$

and

$$ {}^cD^\alpha x_2(t) = \lambda x_2(t) + H\left(t, x_2(t), \int_0^t h(t,s)x_2(s)\mathrm{d}s, \delta_2\right), \; x_2(0) = x_0 \quad (6.5)$$

for $t \in J$, where $H : J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}, x_0 \in \mathbb{R}$ and δ_1, δ_2 are real parameters.

The next theorem deals with the continuous dependence of solutions of Eq. (1.2) on parameters.

Theorem 6.2. *Assume that the function H satisfies the conditions:*

$$|H(t,x,y,\delta_1) - H(t,\bar{x},\bar{y},\delta_1)| \le L_1(|x-\bar{x}| + |y-\bar{y}|), \quad (6.6)$$
$$|H(t,x,y,\delta_1) - H(t,x,y,\delta_2)| \le L_2 \left|\delta_1 - \delta_2\right|, \quad (6.7)$$

where $L_1, L_2 \ge 0$. Let $x_1(t)$ and $x_2(t)$ be the solutions of equation (6.4) and (6.5), respectively. Then,

$$|x_1(t) - x_2(t)| \le |\delta_1 - \delta_2| \, L_2\left[\frac{1}{\Gamma(\alpha+1)}t^\alpha + \frac{1}{\Gamma(\alpha+1)}t^\alpha \int_0^t \frac{(\lambda+L_1)}{\Gamma(\alpha)}(t-s)^{\alpha-1} \right.$$
$$\left. \times exp\left\{\frac{(\lambda+L_1)T^\alpha}{\Gamma(\alpha+1)} + \frac{L_1 h_T T}{(\lambda+L_1)}\right\}\mathrm{d}s\right]. \quad (6.8)$$

Proof. Let $x_1(t)$ and $x_2(t)$ be the solution of (6.4) and (6.5) respectively, then

$$ {}^cD^\alpha x_1(t) = \lambda x_1(t) + H\left(t, x_1(t), \int_0^t h(t,s)x_1(s)\mathrm{d}s, \delta_1\right), \; x_1(0) = x_0$$

and

$$ {}^cD^\alpha x_2(t) = \lambda x_2(t) + H\left(t, x_2(t), \int_0^t h(t,s)x_2(s)\mathrm{d}s, \delta_2\right), \; x_2(0) = x_0$$

This gives

$$x_1(t) = x_0 + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_1(s)\,\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} H\left(s, x_1(s), \int_0^s h(s,\tau)x_1(\tau)\mathrm{d}\tau, \delta_1\right)\mathrm{d}s$$

and

$$x_2(t) = x_0 + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_2(s)\,\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} H\left(s, x_2(s), \int_0^s h(s,\tau)x_2(\tau)\mathrm{d}\tau, \delta_2\right)\mathrm{d}s,$$

Therefore, for any $t \in J$,

$$|x_1(t) - x_2(t)| \le \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|x_1(s) - x_2(s)\right|\,\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigg| H\left(s, x_1(s), \int_0^s h(s,\tau)x_1(\tau)\mathrm{d}\tau, \delta_1\right)$$
$$-H\left(s, x_2(s), \int_0^s h(s,\tau)x_2(\tau)\mathrm{d}\tau, \delta_2\right)\bigg|\,\mathrm{d}s$$

$$
\begin{aligned}
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x_1(s)-x_2(s)| \, \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigg| H\left(s, x_1(s), \int_0^s h(s,\tau)x_1(\tau)\mathrm{d}\tau, \delta_1\right) \\
&\quad - H\left(s, x_2(s), \int_0^s h(s,\tau)x_2(\tau)\mathrm{d}\tau, \delta_1\right)\bigg| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigg| H\left(s, x_2(s), \int_0^s h(s,\tau)x_1(\tau)\mathrm{d}\tau\mathrm{d}\tau, \delta_1\right) \\
&\quad - H\left(s, x_2(s), \int_0^s h(s,\tau)x_2(\tau)\mathrm{d}\tau, \delta_2\right)\bigg| \mathrm{d}s \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x_1(s)-x_2(s)| \, \mathrm{d}s + \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \left(|x_1(s)-x_2(s)| + \int_0^s |h(s,\tau)| \, |x(\tau)-y(\tau)| \, \mathrm{d}\tau \right) \mathrm{d}s \\
&\quad + \frac{L_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\delta_1-\delta_2| \, \mathrm{d}s \\
&\leq \frac{|\delta_1-\delta_2| \, L_2}{\Gamma(\alpha+1)} t^\alpha + \frac{(\lambda+L_1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x_1(s)-x_2(s)| \, \mathrm{d}s \\
&\quad + \frac{L_1 h_T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s |x(\tau)-y(\tau)| \, \mathrm{d}\tau \right) \mathrm{d}s \\
&\leq \frac{|\delta_1-\delta_2| \, L_2}{\Gamma(\alpha+1)} t^\alpha + \int_0^t \frac{(\lambda+L_1)}{\Gamma(\alpha)} (t-s)^{\alpha-1} \bigg[|x_1(s)-x_2(s)| \\
&\quad + \int_0^s \frac{L_1 h_T}{(\lambda+L_1)} |x_1(\tau)-x_2(\tau)| \, \mathrm{d}\tau \bigg] \mathrm{d}s
\end{aligned}
$$

Applying Pachpatte's inequality given in the Theorem 2.8 with

$$
u(t) = |x(t)-y(t)|, \; n(t) = \frac{|\delta_1-\delta_2| \, L_2}{\Gamma(\alpha+1)} t^\alpha, \; f(s) = \frac{(\lambda+L_1)}{\Gamma(\alpha)} (t-s)^{\alpha-1},
$$

$$
q(\tau) = \frac{L_1 h_T}{(\lambda+L_1)},
$$

we obtain

$$
\begin{aligned}
|x_1(t)-x_2(t)| \leq \frac{|\delta_1-\delta_2| \, L_2}{\Gamma(\alpha+1)} t^\alpha \bigg[1 + \int_0^t \frac{(\lambda+L_1)}{\Gamma(\alpha)} (t-s)^{\alpha-1} \\
\times exp\bigg(\int_0^s \bigg\{ \frac{(\lambda+L_1)}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} \\
+ \frac{L_1 h_T}{(\lambda+L_1)} \bigg\} \mathrm{d}\tau \bigg) \mathrm{d}s \bigg]
\end{aligned}
$$

Finally, we get

$$|x_1(t) - x_2(t)| \leq |\delta_1 - \delta_2|\, L_2 \Bigg[\frac{1}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(\alpha+1)} t^{\alpha} \int_0^t \frac{(\lambda + L_1)}{\Gamma(\alpha)} (t-s)^{\alpha-1} exp\Bigg\{ \frac{(\lambda+L_1)T^{\alpha}}{\Gamma(\alpha+1)} + \frac{L_1 h_T T}{(\lambda+L_1)} \Bigg\} \mathrm{d}s \Bigg]$$

□

7. Illustrative Example

In this section, we give an example to illustrate the usefulness of our main results.

We consider

$$\begin{cases} {}^{c}D^{\frac{1}{2}} x(t) = x(t) + \dfrac{x(t)+1}{t^2+9} + \dfrac{1}{9} \displaystyle\int_0^t \frac{1}{(2+t)^2} x(s)\mathrm{d}s, \ t \in [0,1], \\ x(0) = 0. \end{cases} \tag{7.1}$$

Define $f(t, x(t), K_1x(t)) = \dfrac{x(t)+1}{t^2+9} + \dfrac{1}{9} K_1 x(t), \ t \in [0,1], \ \alpha = \dfrac{1}{2}, \ \lambda = 1.$ where $K_1x(t) = \displaystyle\int_0^t \frac{1}{(2+t)^2} x(s)\mathrm{d}s$

Clearly, the function f is continuous.

For any $x_1, x_2 \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t, x_1, K_1x_1) - f(t, x_2, K_1x_2)| \leq \frac{1}{9} \Big[\ |x_1 - x_2| + |K_1x_1 - K_1x_2| \Big]$$

Hence hypothesis (H1) is satisfied with $L = \frac{1}{9}$. It follows from Theorem 3.2 that the problem (7.1) has a unique solution on [0,1].

8. Conclusion

In this paper, we have successfully established an existence and interval of existence of solutions, uniqueness, continuous dependence of the solutions on initial conditions, estimates on solutions and continuous dependence on parameters and functions involved in a nonlinear fractional integro–differential equation with constant coefficient. The important fact of this paper is that with the minimum assumptions on the function f, we have obtained various properties of solutions of a nonlinear fractional integro–differential equation with constant coefficient. In the future, we will extend the results to other fractional derivatives and boundary value problems.

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