

A decorative graphic on the left side of the slide features three balloons: a green one at the top, a light blue one in the middle, and a purple one at the bottom. Each balloon is connected to a streamer that curves downwards. Small yellow triangular shapes are scattered around the balloons, resembling confetti or streamer details.

7.1 Sample space, events, probability

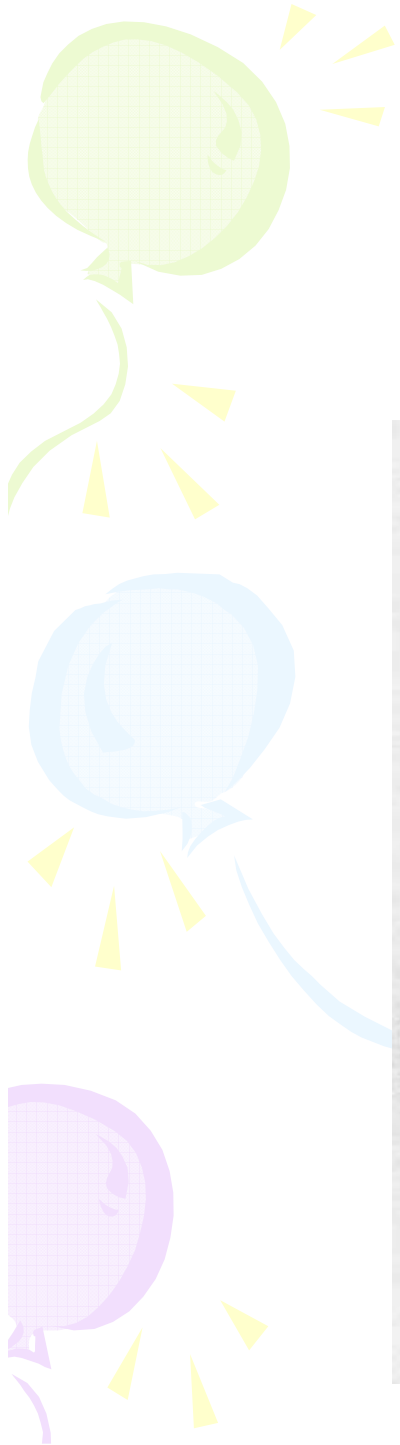
- In this chapter, we will study the topic of probability which is used in many different areas including insurance, science, marketing, government and many other areas.

Blaise Pascal-father of modern probability

<http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Pascal.html>

- **Blaise Pascal**
- **Born: 19 June 1623 in Clermont (now Clermont-Ferrand), Auvergne, France**
- **Died: 19 Aug 1662 in Paris, France**
- In correspondence with [Fermat](#) he laid the foundation for the [theory of probability](#). This correspondence consisted of five letters and occurred in the summer of 1654. They considered the dice problem, already studied by [Cardan](#), and the problem of points also considered by [Cardan](#) and, around the same time, [Pacioli](#) and [Tartaglia](#). The dice problem asks how many times one must throw a pair of dice before one expects a double six while the problem of points asks how to divide the stakes if a game of dice is incomplete. They solved the problem of points for a two player game but did not develop powerful enough mathematical methods to solve it for three or more players.

Pascal





Probability

- 1. Important in **inferential statistics**, a branch of statistics that relies on sample information to make decisions about a population.
- 2. Used to make decisions in the face of uncertainty.

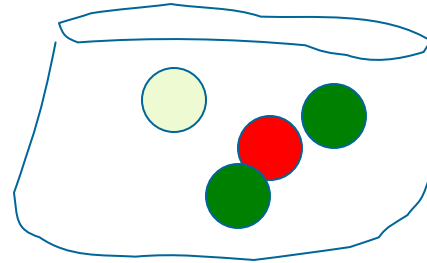
Terminology

- **1. Random experiment**
: is a process or activity which produces a number of possible outcomes. The outcomes which result cannot be predicted with absolute certainty.
- **Example 1:** Flip two coins and observe the possible outcomes of heads and tails

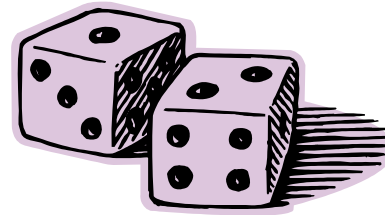


Examples

- 2. Select two marbles without replacement from a bag containing 1 white, 1 red and 2 green marbles.



- 3. Roll two die and observe the sum of the points on the top faces of each die.



- All of the above are considered **experiments**.



Terminology

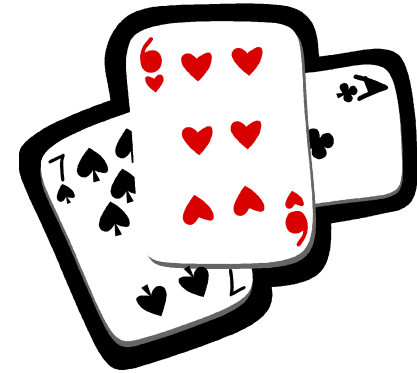
- **Sample space:** is a list of all possible outcomes of the experiment. The outcomes must be mutually exclusive and exhaustive. Mutually exclusive means they are distinct and non-overlapping. Exhaustive means complete.
- **Event:** is a subset of the sample space. An event can be classified as a **simple event** or **compound event**.



Terminology

- 1. Select two marbles in succession without replacement from a bag containing 1 red, 1 blue and two green marbles.
- 2. Observe the possible sums of points on the top faces of two dice.

- 3. Select a card from an ordinary deck of playing cards (no jokers)
- The sample space would consist of the 52 cards, 13 of each suit. We have 13 clubs, 13 spades, 13 hearts and 13 diamonds.

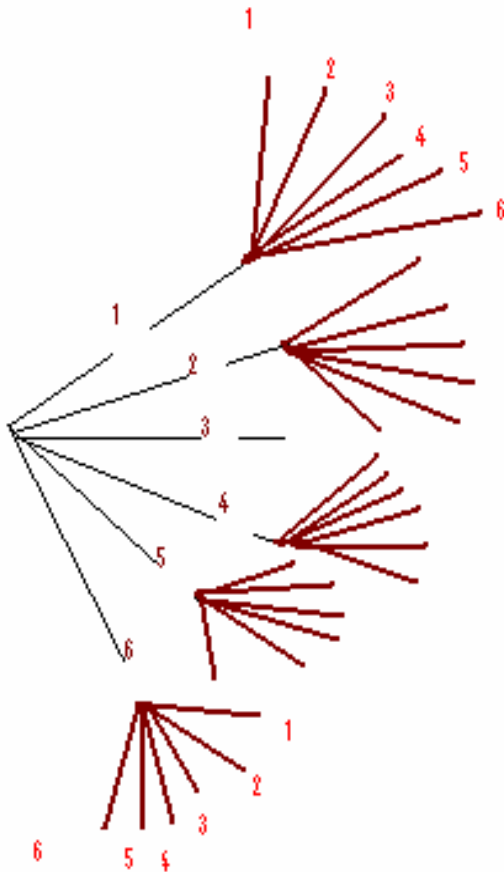


- A **simple event**: the selected card is the two of clubs. A **compound event** is the selected card is red (there are 26 red cards and so there are 26 simple events comprising the compound event)

- 4. Select a driver randomly from all drivers in the age category of 18-25. (Identify the sample space, give an example of a simple event and a compound event)

More examples

- Roll two dice.
- Describe the sample space of this event.



You can use a tree diagram to determine the sample space of this experiment. There are six outcomes on the first die $\{1,2,3,4,5,6\}$ and those outcomes are represented by six branches of the tree starting from the “tree trunk”. For each of these six outcomes, there are six outcomes, represented by the brown branches. By the fundamental counting principle, there are $6 \times 6 = 36$ outcomes. They are listed on the next slide.



Sample space of all possible outcomes
when two dice are tossed.

- $(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
- $(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
- $(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
- $(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
- $(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
- $(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)$
- Quite a tedious project !!



Probability of an event

- Definition: sum of the probabilities of the simple events that constitute the event. The **theoretical probability** of an event is defined as the number of ways the event can occur divided by the number of events of the **sample space**. Using mathematical notation, we have

- $$P(E) = \frac{n(E)}{n(S)}$$
 $n(E)$ is the number of ways the event

- can occur and $n(S)$ represents the total number of events in the sample space.
- 

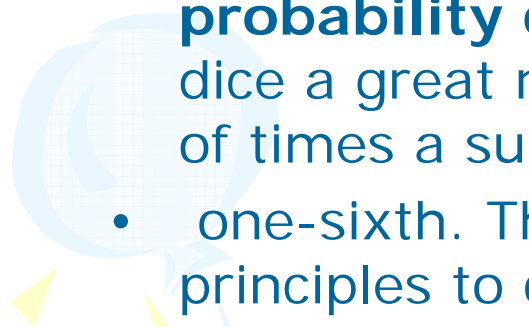
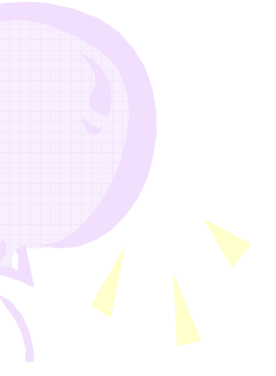


Examples

- Example: Probability of a sum of 7 when two dice are rolled. First we must calculate the number of events of the sample space. From our previous example, we know that there are 36 possible sums that can occur when two dice are rolled. Of these 36 possibilities, how many ways can a sum of seven occur? Looking back at the slide that gives the sample space we find that we can obtain a sum of seven by the outcomes { (1,6), (6,1), (2,5), (5,2), (4,3) (3,4) } There are six ways two obtain a sum of seven. The outcome (1,6) is different from (6,1) in that (1,6) means a one on the first die and a six on the second die, while a (6,1) outcome represents a six on the first die and one on the second die. The answer is $P(E) = \frac{n(E)}{n(S)}$
- $= \frac{6}{36} = \frac{1}{6}$



Meaning of probability

- How do we interpret this result? What does it mean to say that the probability that a sum of seven occurs upon rolling two dice is $1/6$? This is what we call the **long-range probability** or **theoretical probability**. If you rolled two dice a great number of times, in the long run the proportion of times a sum of seven came up would be approximately
 - one-sixth. The **theoretical probability** uses mathematical principles to calculate this probability **without doing an experiment**. The theoretical probability of an event should be close to the **experimental probability** if the experiment is repeated a great number of times.
- 
- 



Some properties of probability

• 1

$$0 \leq p(E) \leq 1$$

• 2.

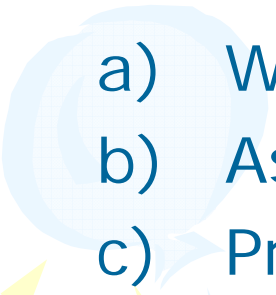
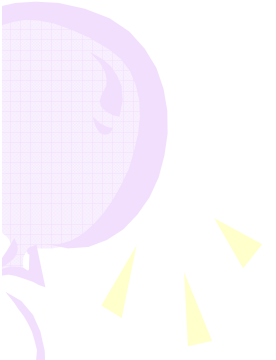
$$P(E_1) + P(E_2) + P(E_3) + \dots = 1$$

- The first property states that the probability of any event will always be a decimal or fraction that is between 0 and 1 (inclusive). If $P(E)$ is 0, we say that event E is an **impossible event**. If $p(E) = 1$, we call event E a certain event. Some have said that there are two certainties in life: death and taxes.
- The second property states that the sum of all the individual probabilities of each event of the sample space must equal one.



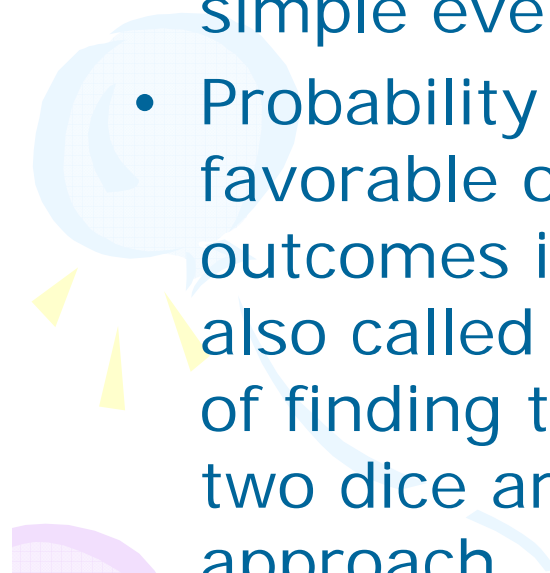
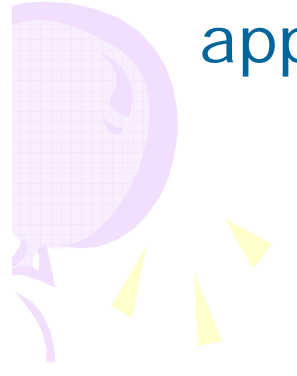
Examples

A quiz contains a multiple-choice question with five possible answers, only one of which is correct. A student plans to guess the answer.

- 
- a) What is sample space?
 - b) Assign probabilities to the simple events
 - c) Probability student guesses the wrong answer
 - d) Probability student guesses the correct answer.
- 



Three approaches to assigning probabilities

- **1. Classical approach.** This type of probability relies upon mathematical laws. Assumes all simple events are equally likely.
 - Probability of an event $E = p(E) = (\text{number of favorable outcomes of } E) / (\text{number of total outcomes in the sample space})$ This approach is also called **theoretical probability**. The example of finding the probability of a sum of seven when two dice are tossed is an example of the classical approach.
- 
- 



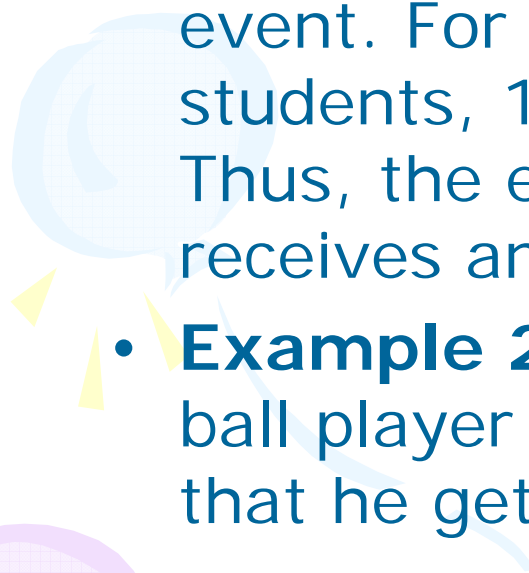
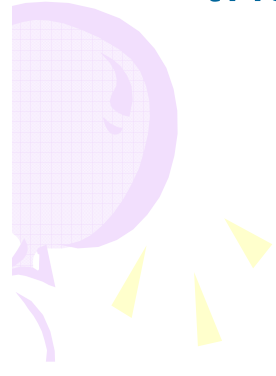
Example of classical probability

- Example: Toss two coins. Find the probability of at least one head appearing.
- Solution: At least one head is interpreted as one head or two heads.
- Step 1: Find the sample space: { HH, HT, TH, TT} There are four possible outcomes.
- Step 2: How many outcomes of the event "at least one head" Answer: 3 : { HH, HT, TH}

- Step 3: Use $P(E) = \frac{n(E)}{n(S)} = \frac{3}{4} = 0.75 = 75\%$

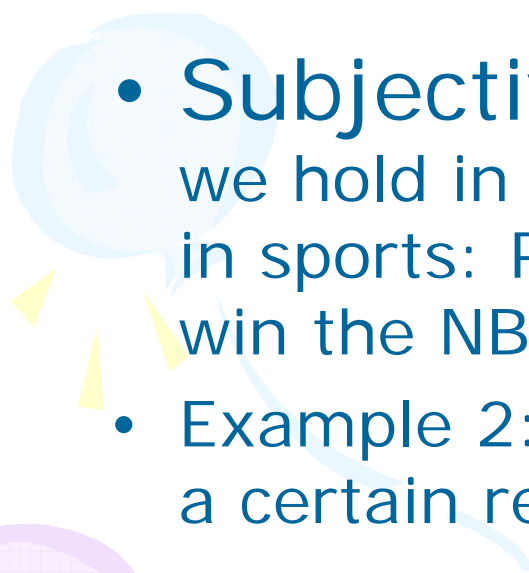
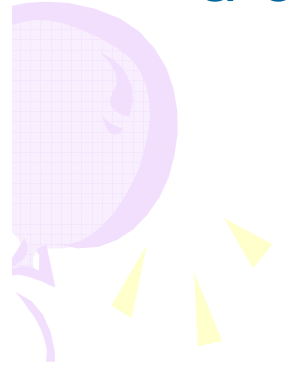


Relative Frequency

- Also called Empirical probability.
 - Relies upon the long run relative frequency of an event. For example, out of the last 1000 statistics students, 15 % of the students received an A. Thus, the empirical probability that a student receives an A is 0.15.
 - **Example 2:** Batting average of a major league ball player can be interpreted as the probability that he gets a hit on a given at bat.
- 
- 



Subjective Approach

- 1. Classical approach not reasonable
 - 2. No history of outcomes.
 - **Subjective approach:** The degree of belief we hold in the occurrence of an event. Example in sports: Probability that San Antonio Spurs will win the NBA title.
 - Example 2: Probability of a nuclear meltdown in a certain reactor.
- 
- 



Example

- The manager of a records store has kept track of the number of CD's sold of a particular type per day. On the basis of this information, the manager produced the following list of the number of daily sales:

Number of CDs	Probability
• 0	0.08
• 1	0.17
• 2	0.26
• 3	0.21
• 4	0.18
• 5	0.10


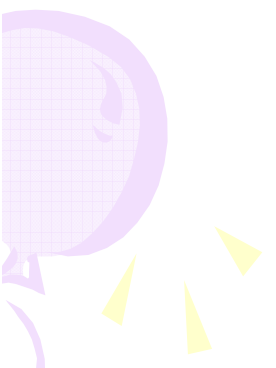


Example continued

- 1. define the experiment as the number of CD's sold tomorrow. Define the sample space
- 2. Prob(number of CD's sold > 3)
- 3. Prob of selling five CD's
- 4. Prob that number of CD's sold is between 1 and 5?
- 5. probability of selling 6 CD's



7.2 Union, intersection, complement of an event, odds

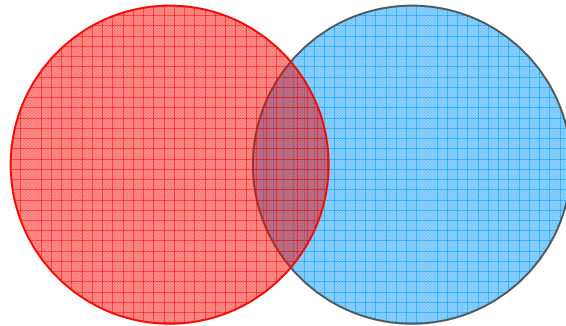
- In this section, we will develop the rules of probability for **compound events** (more than one event) and will discuss probabilities involving the **union** of events as well as **intersection** of two events.
- 
- 

The number of events in the union of A and B is equal to the number in A plus the number in B minus the number of events that are in both A and B.

Sample
space S

$$N(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Event A

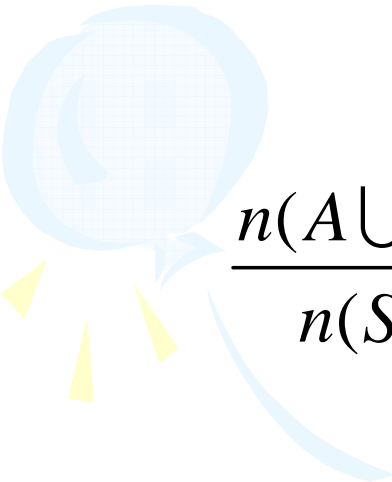


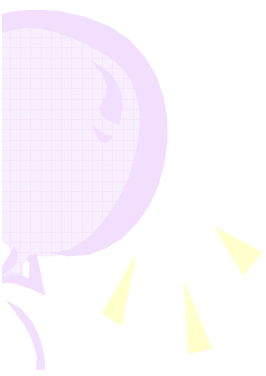
Event B



Addition Rule

- If you divide both sides of the equation by $n(S)$, the number in the sample space, we can convert the equation to an equation of probabilities:

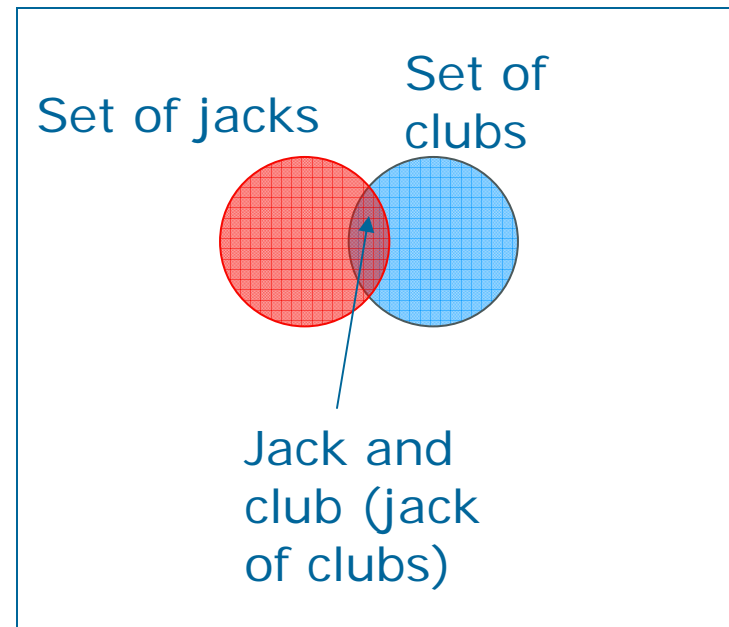

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} \rightarrow$$


$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Addition Rule

- A single card is drawn from a deck of cards. Find the probability that
 - the card is a jack or club.
 - $P(J \text{ or } C) = p(J) + p(C) - P(J \text{ and } C)$

$$\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

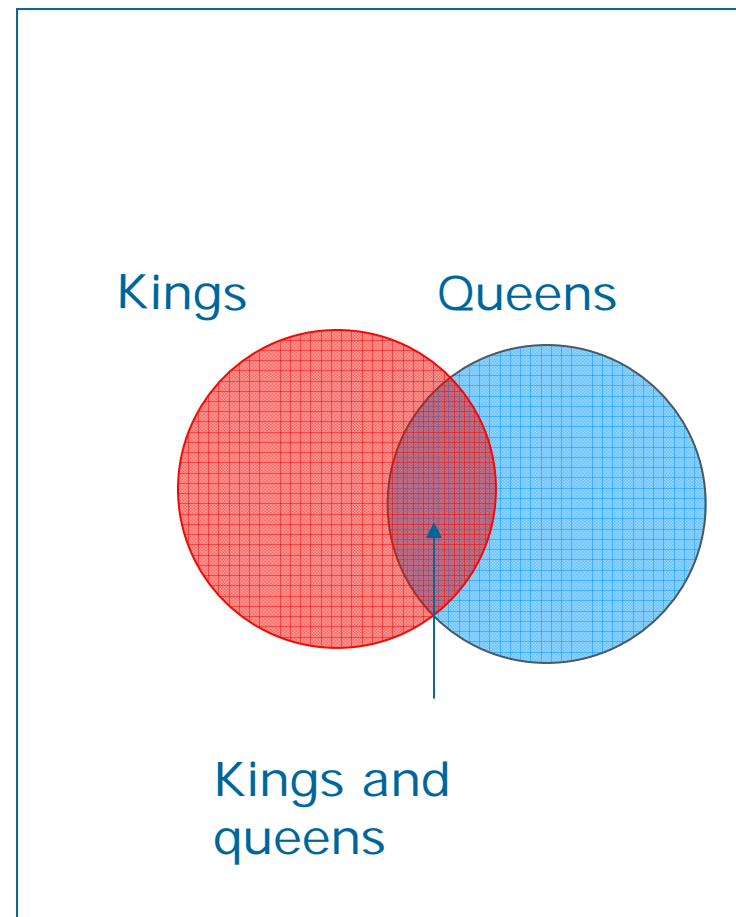


The events King and Queen are mutually exclusive. They cannot occur at the same time. So the probability of a king **and queen** is zero.

- the card is king or queen

$$P(K \cup Q) = p(K) + P(Q) - p(K \cap Q)$$

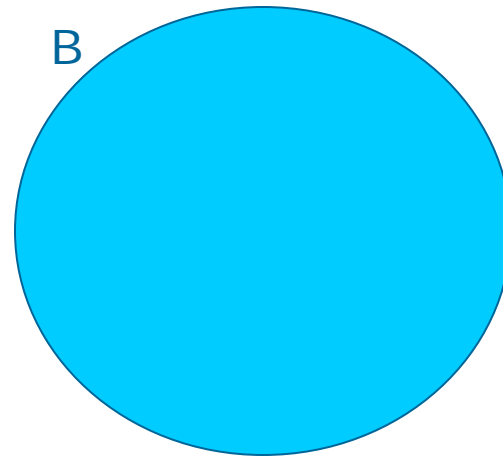
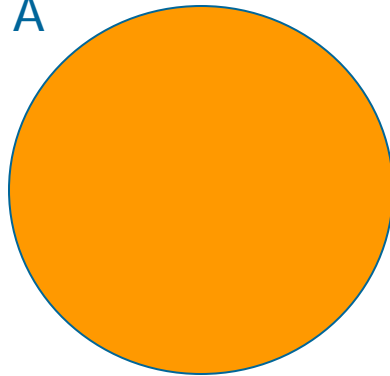
- $= 4/52 + 4/52 - 0 =$
- $8/52 = 2/13$



Mutually exclusive events

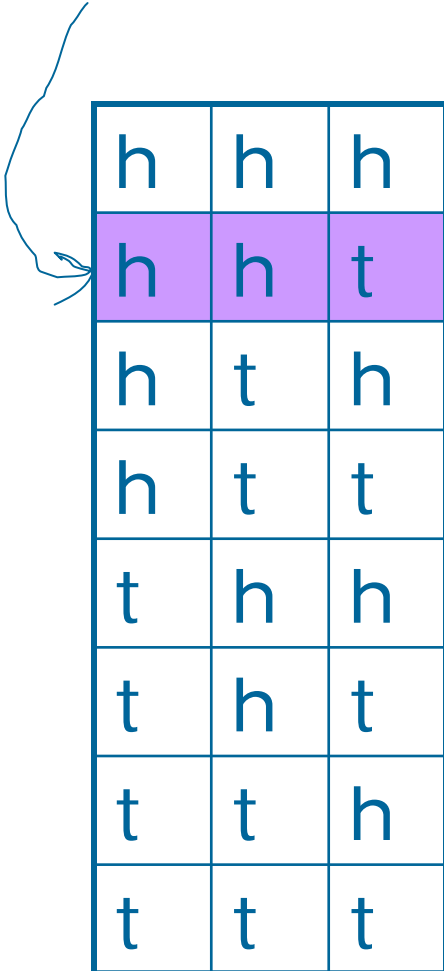
If A and B are mutually exclusive then $P(A \cup B) = p(A) + p(B)$

The intersection of A and B is the empty set

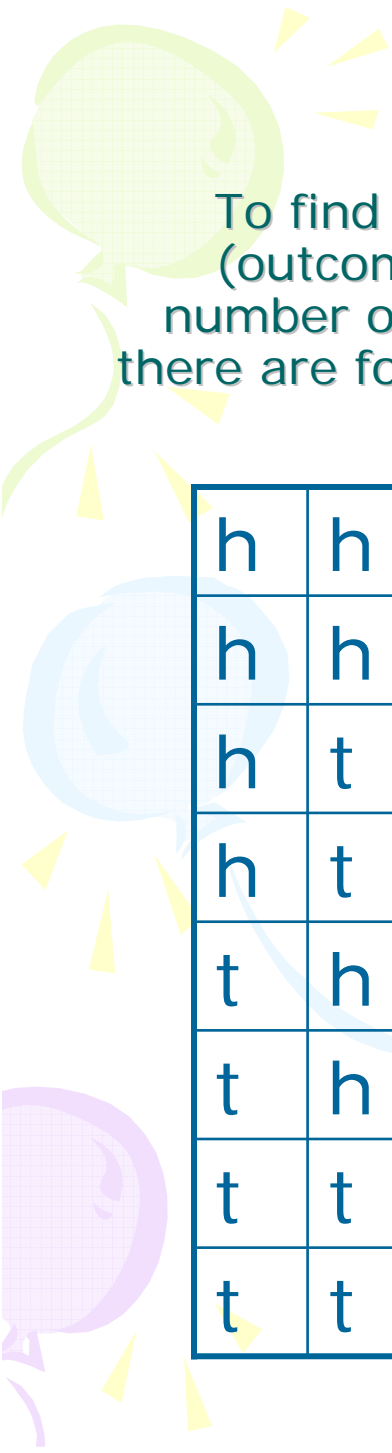


Use a table to list outcomes of an experiment

- Three coins are tossed. Assume they are fair coins. Give the sample space. Tossing three coins is the same experiment as tossing one coin three times. There are two outcomes on the first toss, two outcomes on the second toss and two outcomes on toss three. Use the multiplication principle to calculate the total number of outcomes: $(2)(2)(2)=8$ We can list the outcomes using a little "trick" In the far left hand column, write four H's followed by four T's. In the middle column, we write 2 H's, then two T's, two H's , then 2 T's. In the right column, write T,H,T,H,T,H,T,H . Each row of the table consists of a simple event of the sample space. The indicated row, for instance, illustrates the outcome {heads, heads, tails} in that order.



h	h	h
h	h	t
h	t	h
h	t	t
t	h	h
t	h	t
t	t	h
t	t	t



To find the probability of at least two tails, we mark each row (outcome) that contains two tails or three tails and divide the number of marked rows by 8 (number in the sample space) Since there are four outcomes that have at least two tails, the probability is $\frac{4}{8}$ or $\frac{1}{2}$.

h	h	h
h	h	t
h	t	h
h	t	t
t	h	h
t	h	t
t	t	h
t	t	t



Two dice are tossed. What is the probability of a sum greater than 8 or doubles?

$$P(S > 8 \text{ or doubles}) = p(S > 8) + p(\text{doubles}) - p(S > 8 \text{ and doubles}) = 10/36 + 6/36 - 2/36 = 14/36 = 7/18.$$

- (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)
- (2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
- (3,1), (3,2), (3,3), (3,4), (3,5), (3,6)
- (4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
- (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)
- (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)

Circled elements belong to the intersection of the two events.

Complement Rule

Many times it is easier to compute the probability that A won't occur
then the probability of event A.

$$P(A) + p(\textit{not } A) = 1 \rightarrow$$

$$P(\textit{not } A) = 1 - P(A)$$

- Example: What is the probability that when two dice are tossed, the number of points on each die will **not** be the same?
- This is the same as saying that **doubles** will not occur. Since the probability of doubles is $6/36 = 1/6$, then the probability that doubles **will not occur** is $1 - 6/36 = 30/36 = 5/6$.



Odds

- In certain situations, such as the gaming industry, it is customary to speak of the odds in favor of an event E and the odds against E .

- Definition: **Odds in favor of event E** = $\frac{P(E)}{p(E')}$

- **Odds against E** = $\frac{p(E')}{P(E)}$

- **Example:** Find the odds in favor of rolling a seven when two dice are tossed.

- Solution: The probability of a sum of seven is $6/36$. So

$$\frac{P(E)}{p(E')} = \frac{\frac{6}{36}}{\frac{30}{36}} = \frac{6}{30} = \frac{1}{5}$$

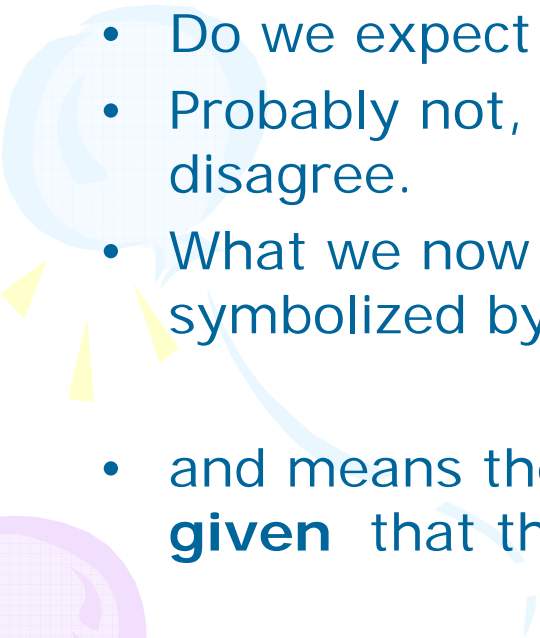
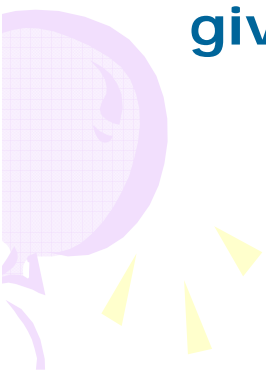
7.3 Conditional Probability, Intersection and Independence

- Consider the following problem:
- Find the probability that a randomly chosen person in the U.S. has lung cancer.
- We want : $p(C)$. To determine the answer, we must know how many individuals are in the sample space, $n(S)$. Of those, how many have lung cancer, $n(C)$ and find the ratio of $n(c)$ to $n(S)$.

$$P(C) = \frac{n(C)}{n(S)}$$



Conditional Probability

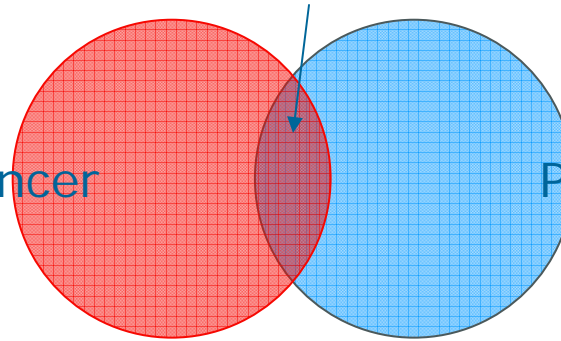
- Now, we will modify the problem: Find the probability that a person has lung cancer, **given** that the person smokes.
 - Do we expect the probability of cancer to be the same?
 - Probably not, although the cigarette manufacturers may disagree.
 - What we now have is called **conditional probability**. It is symbolized by $P(C|S)$
 - and means the probability of lung cancer **assuming or given** that the person smokes.
- 
- 

The probability of having lung cancer given that the person smokes is found by determining the number of people who have lung cancer **and smoke** and dividing that number by the number of smokers.

$$p(L|S) = \frac{n(L \cap S)}{n(S)}$$

people who smoke and have lung cancer.

People with lung cancer



People who smoke



Formula for Conditional probability

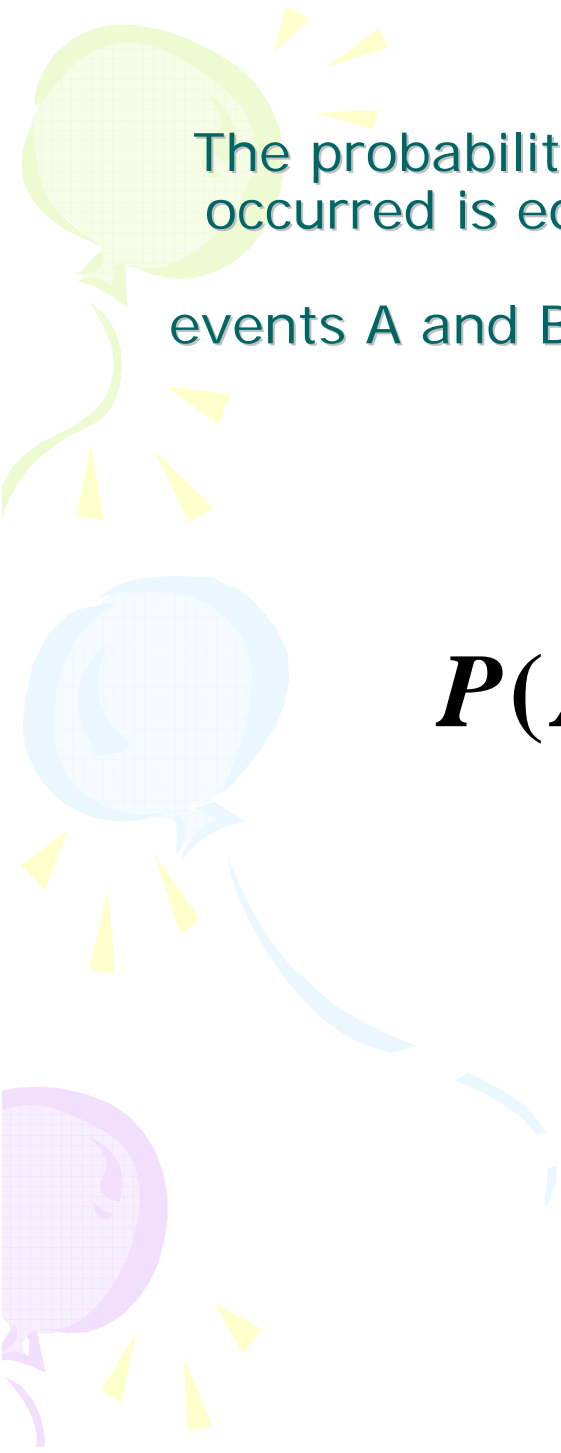
- Derivation:

$$p(L|S) = \frac{n(L \cap S)}{n(S)} \rightarrow$$

$$p(L|S) = \frac{\frac{n(L \cap S)}{n(T)}}{\frac{n(S)}{n(T)}} \rightarrow$$

$$p(L|S) = \frac{p(L \cap S)}{p(S)}$$

- Dividing numerator and denominator by the total number , $n(T)$, of the sample space allows us to express the conditional probability of L given S as the quotient of the probability of L and S divided by the probability of smoker.



The probability of event A given that event B has already occurred is equal to the probability of the intersection of events A and B divided by the probability of event B alone.

$$P(A|B) = \frac{p(A \cap B)}{p(B)}$$



Example

- There are two majors of a particular college: Nursing and Engineering. The number of students enrolled in each program is given in the table. Find the following probabilities by using the following table. The row total gives the total number of each category and the number in the bottom-right cell gives the total number of students. A single student is selected at random from this college. Assuming that each student is equally likely to be chosen, find :

Joint and Marginal Probability

- Table

	Under-grads	Grads	
Nursing	53	47	100
Engineers	37	13	50
	90	60	150

- 1. $\text{Prob}(\text{Nurse}) = 100/150 = 2/3$

- 2. $\text{Prob}(\text{Graduate student}) = 60/150 = 2/5$

- 3. $\text{Probability}(\text{Nurse and Graduate student}) = 47/150$

- 4. $\text{Probability}(\text{Engineering and Grad Student}) = 13/150$



Joint, Marginal and Conditional Probability

- Combinations of simple events

- A and B : symbolism $P(A \cap B) =$

- Probability of intersection is **joint probability**.

- The symbolism represents the probability of the intersection of events A and B.
- 
- 



Conditional probability

- Given that a under-graduate student is selected at random, what is the probability that this student is a nurse?
- Restricting our attention on the column representing undergrads, we find that of the 90 undergrad students, 53 are nursing majors. Therefore, $P(N/U) = 53/90$

	Under-grads	Grads	
Nursing	53	47	100
Engineers	37	13	50
	90	60	150

Using the table

- Given that an engineering student is selected, find the probability that the student is a under-graduate student. Restricting the sample space to the 50 engineering students, 37 of the 50 are undergrads, indicated by the red cell. Therefore, $P(U/E) = 37/50 = 0.74$

	Under-grads	Grads	
Nursing	53	47	100
Engineers	37	13	50
	90	60	150

Derivation of general formulas for P(A and B) using basic algebra

- Algebra:

$$P(B | A) = \frac{P(B \cap A)}{P(A)} \rightarrow$$

$$P(B | A)P(A) = p(B \cap A)$$

$$P(A | B) = \frac{p(A \cap B)}{p(B)}$$

$$P(A | B)p(B) = P(A \cap B)$$

- Since

$$P(A \cap B) = p(B \cap A)$$

- We have

$$\begin{aligned} P(A \cap B) &= p(B \cap A) \\ &= P(A)p(B | A) = p(B)p(A | B) \end{aligned}$$



Two cards are drawn without replacement from an ordinary deck of cards . Find

- Probability (two clubs are drawn in succession).
- Symbolize mathematically:

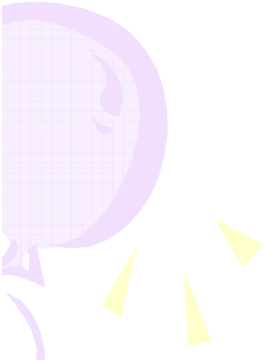
$$P(C_1 \cap C_2)$$



means draw a club on the first draw and then a second club.

$$P(C_1 \cap C_2) = p(C_1) \cdot p(C_2 | C_1)$$

$$= \frac{13}{52} \cdot \frac{12}{51} = \frac{1}{4} \cdot \frac{4}{17} = \frac{1}{17}$$

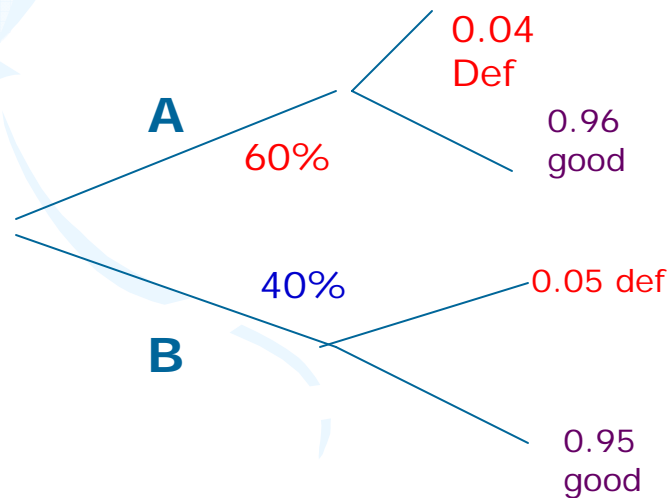
- Because the selection is done without replacement, we have one less card in the sample space and one less club since we **assume** that the first card drawn is a club, there are 12 remaining clubs and 51 total remaining cards.
- 

- Two machines are in operation. Machine A produces 60% of the items whereas machine B produces the remaining 40%. Machine A produces 4% defective items whereas machine B produces 5% defective items. An item is chosen at random.

a) P (item is defective) =
P(D and machine A) or P(D and Machine B)

$$p(D) = P(A \cap D) + p(B \cap D) = p(A)p(D|A) + p(B)p(D|B)$$

$$= 0.60(0.04) + 0.40(0.05) = 0.044$$





Independence

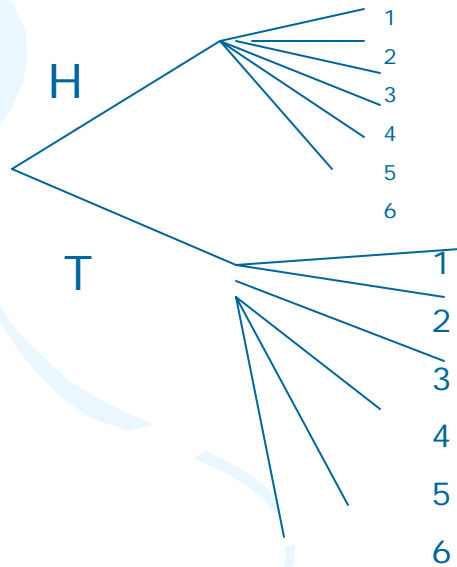
- If two events are independent, then

$$p(A | B) = p(A)$$

$$P(B | A) = p(B)$$

A coin is tossed and a die is rolled. Find the probability that the coin comes up heads and the die comes up three.

- The number of outcomes for the coin is 2: { H , T}. The number of outcomes for the die is 6: { 1,2,3,4,5,6} . Using the fundamental principle of counting, we find that there are $2(6)=12$ total outcomes of the sample space.



$$p(\text{H and 3}) = 1/12$$

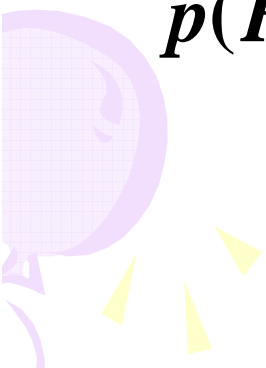


Now, let's look at the same problem in a slightly different way

- To find the probability of Heads and then a three on a dice, we have

$$p(H \cap 3) = p(H) \cdot p(3|H)$$

- using the rule for conditional probability. However, the probability of getting a three on the die **does not depend upon the outcome of the coin toss**. We say that these two events are independent, since the outcome of either one of them does not affect the outcome of the remaining event.


$$p(H \cap 3) = p(H) \cdot p(3|H) = p(H) \cdot p(3)$$



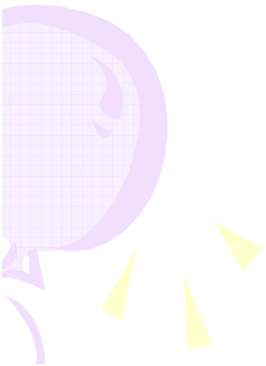
Joint Probability Rule

- If events A and B are independent:

$$P(A \cap B) = P(A) \cdot P(B)$$

- 
- If events A and B are dependent:

$$p(A \cap B) = p(A) \cdot p(B | A)$$



Examples of Independence

- 1. Two cards are drawn in succession **with replacement** from a standard deck of cards. What is the probability that two kings are drawn?

$$\begin{aligned} P(K_1 \cap K_2) &= p(K_1) \cdot p(K_2) \\ &= \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169} \end{aligned}$$

- 2. Two marbles are drawn **with replacement** from a bag containing 7 blue and 3 red marbles. What is the probability of getting a blue on the first draw and a red on the second draw?

$$\begin{aligned} p(B \cap R) &= p(B) \cdot p(R) \\ &= \frac{7}{10} \cdot \frac{3}{10} = \frac{21}{100} = 0.21 \end{aligned}$$

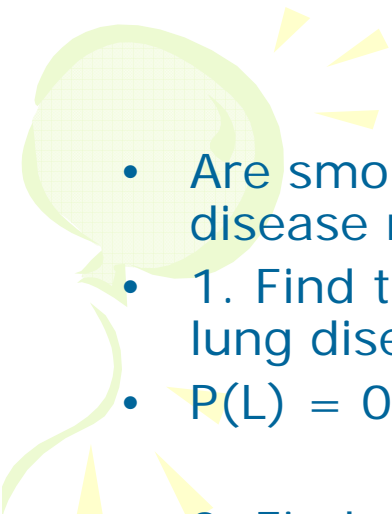


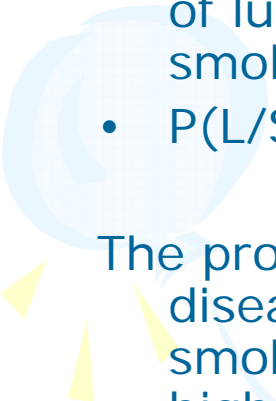
Dependent Events

- Two events are dependent when the outcome of one event affects the outcome of the second event.
-
- Example: Draw two cards in succession without replacement from a standard deck. Find the probability of a king on the first draw and a king on the second draw.


• Answer:

$$\begin{aligned} P(K_1 \cap K_2) &= p(K_1) \cdot p(K_2 | K_1) \\ &= \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221} \end{aligned}$$

- 
- Are smoking and lung disease related?
 - 1. Find the probability of lung disease.
 $P(L) = 0.15$ (row total)
 - 2. Find $p(L/S)$, probability of lung disease given smoker.
 $P(L/S) = 0.12/0.19 = 0.63$.



The probability of having lung disease given that you are a smoker is considerably higher than the probability of lung disease in the general population, so we cannot say that smoking and lung disease are independent events.



	Smoker	Non-smoker
Has Lung Disease	0.12	0.03
No lung Disease	0.19	0.66

The background features several large, overlapping, semi-transparent swirls in shades of purple, green, and blue. Scattered throughout are numerous small, yellow, triangular shapes, some pointing upwards and others downwards, resembling a starburst or confetti effect.

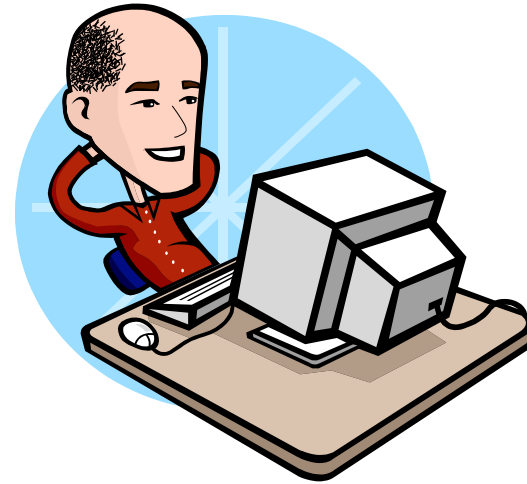
7.4 Baye's Formula

In this section, we will find the probability of an earlier event conditioned on the occurrence of a later event.

Probability of an Earlier Event Given a Later Event

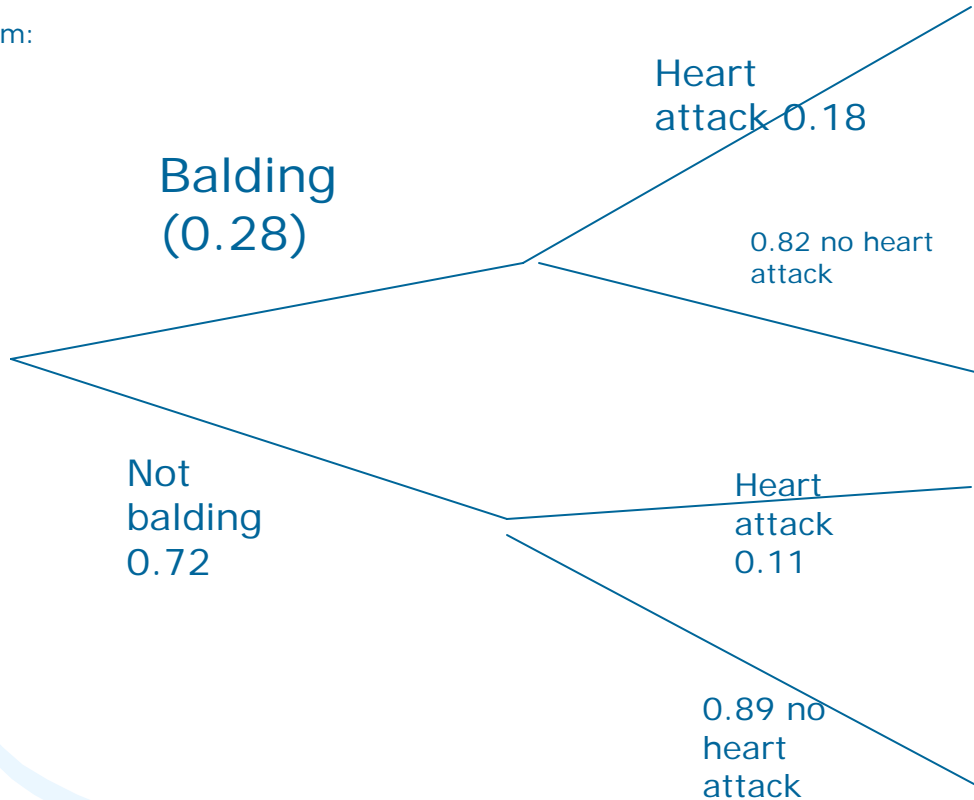
The procedures used in this example can be employed for other problems using Bayesian probability.

- A survey of middle-aged men reveals that 28% of them are balding at the crown of their head. Moreover, it is known that such men have an 18% probability of suffering a heart attack in the next 10 years. Men who are not balding in this way have an 11% probability of a heart attack. If a middle-aged man is randomly chosen, what is the probability that he is balding given that he suffered a heart attack? See tree diagram- next slide.



We want $P(B/H)$ (probability that he is balding given that he suffered a heart attack).

• Tree diagram:



Using the rule for conditional probability, we derive the general formula to solve the problem:

$$P(B|H) = \frac{p(B \cap H)}{p(H)}$$

$$p(H) = p(B \cap H) + p(NB \cap H)$$

$$p(H) = p(B)p(H|B) + p(NB) \cdot p(H|NB)$$

$$P(B|H) = \frac{p(B \cap H)}{p(B)p(H|B) + p(NB) \cdot p(H|NB)}$$

- P(H) can be found by finding the union of
- P(B and H) and
- p(NB and H)

- Rule for conditional probability

- Substitution for p(H)

Using the tree diagram, substitute the probabilities in the formula .
The answer reveals that the probability that a middle-aged man is balding given he suffered a heart attack is 0.389.

1.
$$P(B|H) = \frac{p(B \cap H)}{p(B)p(H|B) + p(NB) \cdot p(H|NB)}$$

2.
$$p(B|H) = \frac{p(B) \cdot p(H|B)}{p(B)p(H|B) + p(NB) \cdot p(H|NB)}$$

3.
$$p(B|H) = \frac{0.28 \cdot (0.18)}{0.28(0.18) + 0.72 \cdot (0.11)}$$

4.
$$= 0.389$$