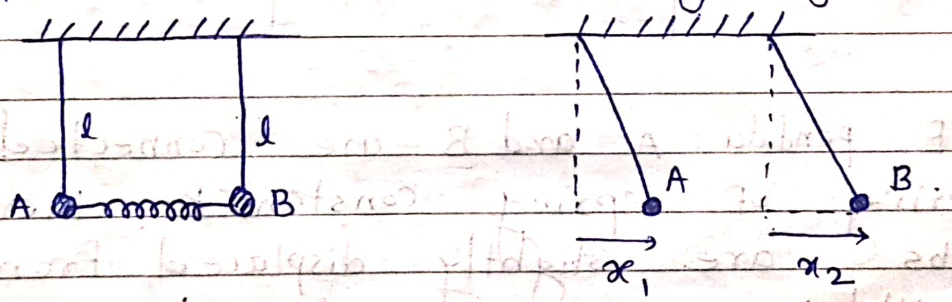
2. Coupled Oscillations

Introduction :- Coupled oscillators are oscillators connected in such away that energy can be transferred between them.

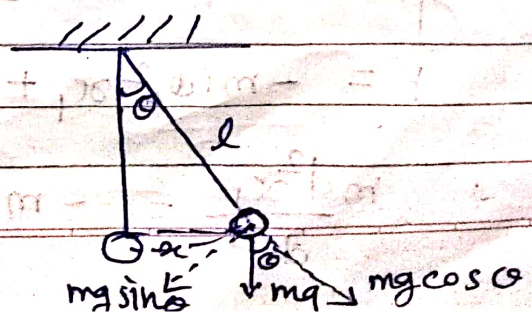
each oscillator oscillates with a well defined frequency called normal mode. These normal modes of a system are its natural frequencies. For any oscillating system natural frequencies depend upon the shape, size, structure and boundary conditions.

The most general motion of a system is a superposition of its normal modes are called harmonic or overtones. Different modes can be excited quite independently. The vibrations of atoms or ions in a solid are the most important and ~~ex~~ examples of coupled oscillations.

* Frequencies of coupled oscillatory systems.



Consider two simple pendula A and B of length (l) each and suspended from a rigid support. Let, m be the mass of each pendulum restoring force acting on the bob for small angular displacement θ is,



$$F = -mg \sin\theta$$

but θ is very small.

$$F = -mg\theta$$

but, $\theta = \frac{\text{arc}}{\text{length}} = \frac{x}{L}$

$$F = -\frac{mgx}{l}$$

Angular frequency of each pendulum when they oscillates independently is,

$$\omega_0 = \sqrt{\frac{g}{L}} \quad \text{or} \quad \omega_0^2 = \frac{g}{l}$$

$\therefore$ Restoring force $= -m\omega_0^2 x$ .. $(\omega_0^2 =$

$$\therefore \boxed{F = -m\omega_0^2 x}$$

If pendula A and B are connected by a spring of spring constant K then the bobs are slightly displaced from the equillibrium position, the two pendula begin to oscillate.

If x_1 and x_2 are the displacements of the pendula at any time(t), then depending on $x_1 > x_2$ or $x_1 < x_2$.

$$F = -m\omega_0^2 x_1 + K(x_2 - x_1)$$

$$\therefore m\frac{d^2x_1}{dt^2} = -m\omega_0^2 x_1 + K(x_2 - x_1)$$

and $m\frac{d^2x_2}{dt^2} = -m\omega_0^2 x_2 - K(x_2 - x_1)$

Now, by putting $\frac{K}{m} = \omega^2$ and simplyfy

we get, $\frac{d^2x_2}{dt^2} = -\omega_0^2 x_2 - \omega^2(x_2 - x_1)$

$$= -\omega_0^2 x_2 - \omega^2 x_2 + \omega^2 x_1$$

$$\frac{d^2x_2}{dt^2} = -(\omega_0^2 + \omega^2)x_2 + \omega^2 x_1 \quad \text{(2)}$$

and $\frac{d^2x_1}{dt^2} = -\omega_0^2 x_1 + \omega^2(x_2 - x_1)$

$$= -\omega_0^2 x_1 + \omega^2 x_2 - \omega^2 x_1$$

$$\frac{d^2x_1}{dt^2} = -(\omega_0^2 + \omega^2)x_1 + \omega^2 x_2 \quad \text{(1)}$$

eq^n (1) & (2) are coupled they involve both variables x_1 and x_2.

motion of one pendulum is affected by the motion of other.

ie. the pendula A & B are coupled.

To solve the above differential equations we have to decouplet the equations.

- Let us, add eq^n (I) & (II).

$$\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} = -(\omega_0^2 + \omega^2)x_1 + \omega^2 x_2 + -(\omega_0^2 + \omega^2)x_2 + \omega^2 x_1$$

$$\frac{d^2(x_1 + x_2)}{dt^2} = -\omega_0^2 x_1 - \cancel{\omega^2 x_1} + \cancel{\omega^2 x_2} - \omega_0^2 x_2 - \cancel{\omega^2 x_2} + \cancel{\omega^2 x_1}$$

$$= -\omega_0^2(x_1 + x_2) \quad \text{(3)}$$

~~if subtract eqn ① & ②.~~

$$\cancel{\frac{d^2}{dt^2}(x_1-x_2) = -\omega_0^2x_1 - \omega^2x_1 + \omega^2x_2 + \omega_0^2x_2 + \omega^2x_2 \pm \omega^2x_1}$$

$$\cancel{= -\omega_0^2(x_1+x_2) + 2\omega^2x_2}$$

$$\cancel{= -\omega_0^2x_1 - \omega_0^2x_2 + 2\omega^2x_2}$$

$$\cancel{= -\omega_0^2x_1 - \omega_0^2x_2 + \omega^2x_2 + \omega^2x_2}$$

$$\frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2} = -(\omega_0^2+\omega^2)x_1 + \omega^2x_2 - \left(-(\omega_0^2+\omega^2)x_2 + \omega^2x_1\right)$$

$$\frac{d^2(x_1-x_2)}{dt^2} = -\omega_0^2x_1 - \omega^2x_1 + \omega^2x_2 - \left(-\omega_0^2x_2 - \omega^2x_2 + \omega^2x_1\right)$$

$$= -\omega_0^2x_1 - \omega^2x_1 + \omega^2x_2 + \omega_0^2x_2 + \omega^2x_2 - \omega^2x_1$$

$$= -\omega_0^2(x_1-x_2) + 2\omega^2x_2 - 2\omega^2x_1$$

$$= -\omega_0^2(x_1-x_2) + 2\omega^2(x_2-x_1)$$

$$= -(\omega_0^2+2\omega^2)(x_1-x_2)$$

$$= -\omega_0^2(x_1-x_2) - 2\omega^2(x_1-x_2)$$

$$= -(\omega_0^2+2\omega^2)(x_1-x_2) \quad \text{——} \quad (4)$$

If we choose new co-ordinates as,

$q_1 = x_1 + x_2$ and $q_2 = x_1 - x_2$ then,

equations (3) and (4) reduce to simple forms,

$$\boxed{\frac{d^2q_1}{dt^2} = -\omega_0^2 q_1} \quad \text{——} \quad (5)$$

and $$\boxed{\frac{d^2 q_2}{dt^2} = -(\omega_0^2 + 2\omega^2)\, q_2} \quad \text{--- (6)}$$

equations (5) & (6) represent simple harmonic motions with frequencies,

$$\omega_1 = \omega_0 = \sqrt{\frac{g}{l}} \quad \text{and} \quad \omega_2 = \sqrt{\omega_0^2 + 2\omega^2}$$

where, $\omega^2 = \frac{k}{m}$.

Thus Coupled pendula oscillate simple harmonically with two frequencies ω_1 and ω_2 which are known as normal modes. and q_1 and q_2 which are Called normal co-ordinates.

* Normal Modes and Normal Co-ordinates.

when a Coupled system oscillates in a single mode, then oscillating parts have same frequency and phase constant. In given mode the relative displacements are governed by a Characteristic Constant ratio of amplitudes.

The solutions of diff. eqⁿs (5) & (6).

$$\therefore q_1 = A_1 \cos\omega_1 t$$

$$\text{and} \quad q_2 = A_2 \cos\omega_2 t$$

$\therefore$ Therefore, in terms of the original coordinates x_1 and x_2 the solutions becomes,

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$$x_1 = \frac{1}{2}(q_1+q_2)$$

$$= \frac{1}{2}(A_1\cos\omega_1 t + A_2\cos\omega_2 t) \quad \text{——} (7)$$

and $$x_2 = \frac{1}{2}(q_1-q_2)$$

$$= \frac{1}{2}(A_1\cos\omega_1 t - A_2\cos\omega_2 t) \quad \text{——} (8)$$

Case-I)

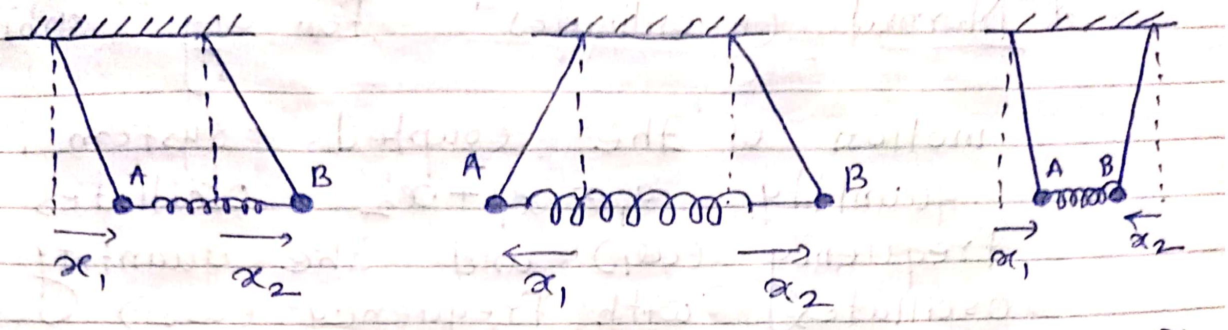
If $A_2 = 0$ then eqⁿ (7) & (8) reduce to

$$x_1 = \frac{A_1}{2}\cos(\omega_1 t)$$

and $$x_2 = \frac{A_1}{2}\cos(\omega_1 t)$$

$$\therefore \; x_1 = x_2$$

In this Case both pendula oscillate with same frequency $\omega_1 = \omega_0 = \sqrt{\frac{g}{l}}$ and they oscillate in phase. both pendula move to left and then to the right simultaneously. In this Case the spring is neither stretched nor Compressed ie. both pendula oscillates independently with same frequency $\omega_1 = \omega_0$ this Called a normal mode.
This normal mode is also known as antisymmetric mode.



$-x_1 = x_2$ $\qquad$ $x_1 = -x_2$

Case - II :- If $A_1 = 0$ then from eqⁿ ⑦ & ⑧

we get, $x_1 = \frac{A_2}{2} \cos\omega_2 t$

and $x_2 = -\frac{A_2}{2} \cos\omega_2 t$

$\therefore \quad x_1 = -x_2$

Thus both pendula oscillates with same frequency.

$$\omega_2 = \left[\omega_0^2 + 2\omega^2\right]^{1/2}$$

$$\omega_2 = \sqrt{\frac{g}{l} + 2\frac{K}{m}}$$

But, they are <u>out of phase</u>.

both pendula move either outwards or inwards when the spring is resp. streched or Compressed. It is also known as symmetric mode.

$\omega_2 > \omega_1$ ie. the frequency of symmetric mode is higher than that of antisymmetric mode. antisymmetric mode (ω_1) is also Called slow oscillation while symmetric mode (ω_2) is known as fast mode.

* **Normal Co-ordinates :-** For any arbitrary motion of the coupled system, the quantity $q_1 = x_1 + x_2$ oscillates with frequency (ω_1) and the quantity $q_2 = x_1 - x_2$ oscillates with frequency (ω_2) where x_1 and x_2 are the displacements of the two pendula at any time (t). These coordinates. These normal co-ordinates q_1 and q_2 are associated with the normal modes ω_1 and ω_2 resp.

* Energy of Two Coupled Oscillators [Beats and Energy Transfer in the system.

When two simple pendula are coupled together by using a spring of spring constant (K) the coupled system oscillates with two normal modes of frequencies given by,

$$\omega_1^2 = \frac{g}{l} \quad \text{and} \quad \omega_2^2 = \left[\omega_0^2 + 2\omega^2\right] = \left[\frac{g}{l} + \frac{2K}{m}\right]$$

in normal modes, the displacements x_A and x_B of the two pendula are obtained by the superposition of two modes.

ie.

$x_A = x_1 + x_2$ For pendulum A

and $x_B = x_1 - x_2$ For pendulum B.

$\therefore x_A = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t$.

and $x_B = A_1 \cos\omega_1 t - A_2 \cos\omega_2 t$.

$\therefore A_1 = A_2 = A$. As two pendula are identical

$\therefore$ Displacements for pendula A & B are,

$$\therefore x_A = A[\cos\omega_1 t + \cos\omega_2 t] \quad \text{--- (1)}$$

$$\text{and} \quad x_B = A[\cos\omega_1 t - \cos\omega_2 t] \quad \text{--- (2)}$$

$\therefore$ velocities of two pendula are given by.

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$$\frac{dx_A}{dt} = A[-\sin\omega_1 t\,(\omega_1) \pm (\sin\omega_2 t)(\omega_2)]$$
$$= -A[\omega_1 \sin\omega_1 t + \omega_2 \sin\omega_2 t]$$

and $$\frac{dx_B}{dt} = A[\omega_1(-\sin\omega_1 t) - (-\sin\omega_2 t)(\omega_2)]$$
$$= -A[\omega_1 \sin\omega_1 t - \omega_2 \sin\omega_2 t]$$

$\therefore$ At any instant of time $t=0$.

$\therefore x_A = 2A$

$\therefore x_B = 0$

$\therefore \frac{dx_A}{dt} = 0$

and $\frac{dx_B}{dt} = 0$

This means both pendula have zero velocity (at rest) and the bob A is given a displacement (2A) and bob B is at its mean position, when both the pendulum bobs are released at $t=0$.

It is observed that the amplitude of pendulum A decreases and at the same time amplitude of B increases, so that A comes to rest when B has maximum displacement. energy is transferred from A to B.

this process repeats when energy is transferred from pendulum B to pendulum A etc.

one round of energy transfer from A to B and back from B to A - is called a beats.

Reciprocal of beat period is beat frequency.

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Since, $\omega_2 > \omega_1$

$$\frac{\omega_2 + \omega_1}{2} = \Omega \quad \text{and} \quad \frac{\omega_2 - \omega_1}{2} = \epsilon$$

Then, $\Omega - \epsilon$

$$\omega_1 = \frac{\omega_2 + \omega_1}{2} - \frac{\omega_2 - \omega_1}{2}$$

and $$\omega_2 = \frac{\omega_2 + \omega_1}{2} + \frac{\omega_2 - \omega_1}{2}$$

$$= \Omega + \epsilon$$

from eqn (1) & (2) we get,

$$x_A = A[\cos(\Omega - \epsilon)t + \cos(\Omega + \epsilon)t]$$

$$= A[\cancel{2\cos}\ \cos\Omega t \cos\epsilon t + \cancel{\sin\Omega t \sin\epsilon t} + \cos\Omega t \cdot \cos\epsilon t - \cancel{\sin\Omega t \sin\epsilon t}]$$

$$= A[2\cos\Omega t \cdot \cos\epsilon t]$$

$$= [2A\cos\epsilon t]\cos\Omega t.$$

$$x_A = A_\epsilon \cos\Omega t \quad \text{———} \quad (3)$$

and $$x_B = A[\cos(\Omega - \epsilon)t - \cos(\Omega + \epsilon)t]$$

$$= A[(\cancel{\cos\Omega t \cdot \cos\epsilon t} + \sin\Omega t \cdot \sin\epsilon t) - (\cancel{\cos\Omega t \cos\epsilon t} - \sin\Omega t \cdot \sin\epsilon t)]$$

$$= A[2\sin\Omega t \cdot \sin\epsilon t]$$

$$= [2A\sin\epsilon t] \cdot \sin\Omega t.$$

$$x_B = B_\epsilon \sin\Omega t. \quad \text{———} \quad (4)$$

where, $A_\epsilon = 2A\cos\epsilon t$ and $B_\epsilon = 2A\sin\epsilon t$. are the amplitude functions which shows a periodic variation.

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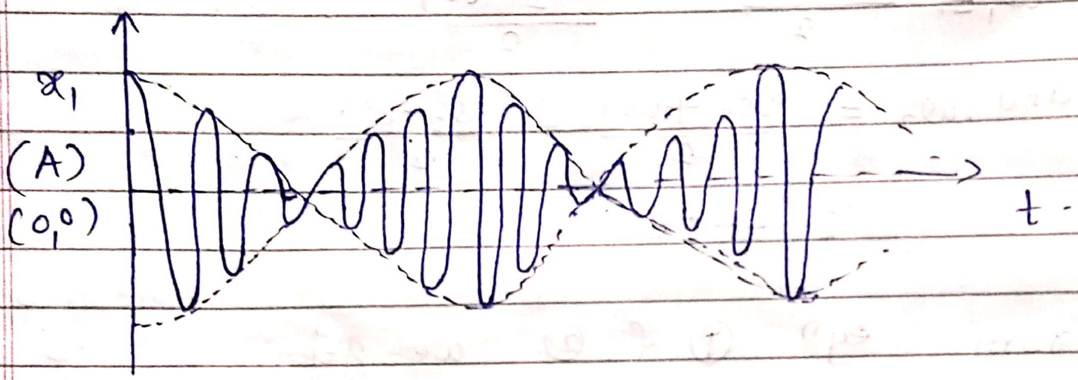
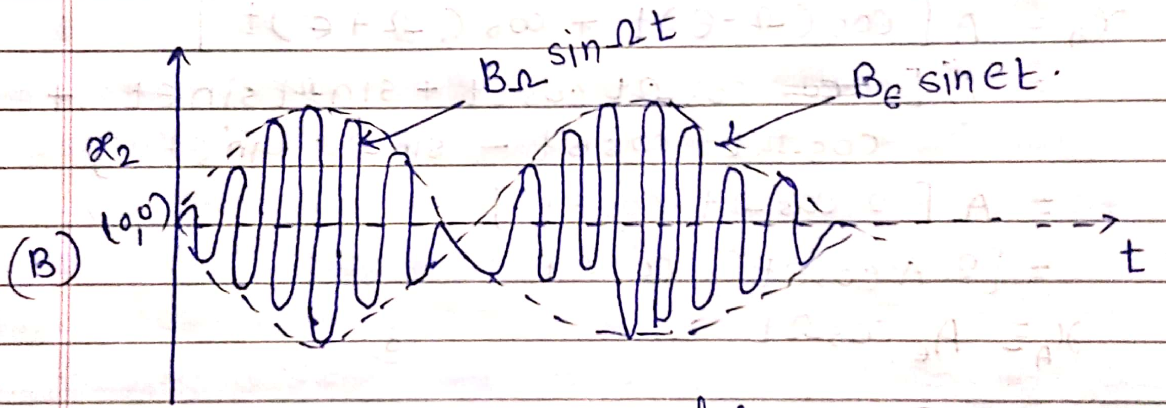
If a weak spring with small spring constant (k) $\omega_1 \simeq \omega_2$ $\epsilon << \Omega$.

ie. ϵt oscillation is much slower as compared to Ωt oscillation. so from eqn (3) & (4) at time instant $t=0$ we get,

$x_A = 2A$. and $x_B = 0$.





oscillations of pendula A & B. with normal mode.

Due to energy transfer beats are produced.

if $\epsilon < 10$ then in case of sound waves beats are heard.

In one beat cycle, pendulum A is considered as a harmonic oscillator with angular frequency Ω with an amplitude $A_\epsilon = 2A\cos\epsilon t$.

$\therefore$ Average K.E. of pendulum A is,

$$E_A = \frac{1}{2}m\Omega^2 A_\epsilon^2$$

$$= \frac{1}{\cancel{2}}m\Omega^2 \overset{2}{\cancel{4}}A^2\cos^2\epsilon t$$

$$= 2mA^2\Omega^2\cos^2\epsilon t$$

Similarly, average K.E. of pendulum B is,

$$E_B = \frac{1}{2}m\Omega^2 B_\epsilon^2$$

$$= \frac{1}{\cancel{2}}m\Omega^2 \overset{2}{\cancel{4}}A^2\sin^2\epsilon t$$

$$= 2mA^2\Omega^2\sin^2\epsilon t$$

$\therefore$ Total energy of coupled system of pendula A and B is,

$$E = E_A + E_B$$

$$\therefore E = 2mA^2\Omega^2\cos^2\epsilon t + 2mA^2\Omega^2\sin^2\epsilon t$$

$$\therefore E = 2mA^2\Omega^2(\cos^2\epsilon t + \sin^2\epsilon t)$$

$$\therefore E = 2mA^2\Omega^2$$

where, $\Omega = \frac{\omega_1 + \omega_2}{2}$

$$\therefore E = \text{constant}$$

Thus total energy is transferred back and forth between two pendula with beat frequency $[2\epsilon = \omega_2 - \omega_1]$