Laplace Transform

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The French Newton
Pierre-Simon Laplace

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Focused later on celestial mechanics
- One of the first scientists to suggest the existence of black holes

History of the Transform

- Euler began looking at integrals as solutions to differential equations in the mid 1700’s:

$$z = \int X(x) e^{ax}\, dx \qquad z(x; a) = \int_0^x e^{at}\, X(t)\, dt\,,$$

- Lagrange took this a step further while working on probability density functions and looked at forms of the following equation:

$$\int X(x) e^{-ax} a^x\, dx,$$

- Finally, in 1785, Laplace began using a transformation to solve equations of finite differences which eventually lead to the current transform

$$S = A y_s + B\, \Delta y_s + C\, \Delta^2 y_s + \ldots, \qquad y_s = \int e^{-sx}\, \phi(x)\, dx\,,$$

Definition

- The Laplace transform is a linear operator that switched a function f(t) to F(s).
- Specifically: $F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} e^{-st} f(t)\,dt.$

  where: $s = \sigma + i\omega.$
- Go from time argument with real input to a complex angular frequency input which is complex.

Restrictions

- There are two governing factors that determine whether Laplace transforms can be used:
  - f(t) must be at least piecewise continuous for $t \geq 0$
  - $|f(t)| \leq Me^{\gamma t}$ where M and γ are constants

Continuity

- Since the general form of the Laplace transform is:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

  it makes sense that f(t) must be at least piecewise continuous for t ≥ 0.
- If f(t) were very nasty, the integral would not be computable.

Boundedness

- This criterion also follows directly from the general definition:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

- If f(t) is not bounded by $Me^{\gamma t}$ then the integral will not converge.

Laplace Transform Theory

- General Theory

$$F(s) = \mathcal{L}\big(f(t)\big) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{\tau\to\infty} \int_0^\tau e^{-st} f(t)\,dt$$

- Example

$$f(t) \equiv 1$$

$$\mathcal{L}\big(f(t)\big) = \int_0^\infty e^{-st}\,1\,dt = \lim_{\tau\to\infty} \left(\frac{e^{-st}}{-s} \Bigg|_0^\tau \right)$$

$$= \lim_{\tau\to\infty} \left(\frac{e^{-s\tau}}{-s} + \frac{1}{s} \right) = \frac{1}{s}$$

- Convergence

$$f(t) \equiv e^{t^2}$$

$$\mathcal{L}\big(f(t)\big) = \lim_{\tau\to\infty} \int_0^\tau e^{-st} e^{t^2}\,dt = \lim_{\tau\to\infty} \int_0^\tau e^{t^2-st}\,dt = \infty$$

Laplace Transforms

- Some Laplace Transforms
- Wide variety of function can be transformed

- Inverse Transform

$$\mathcal{L}^{-1}\big(F(s)\big) = f(t)$$

- Often requires partial fractions or other manipulation to find a form that is easy to apply the inverse

TABLE 6.2.1 Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$

Laplace Transform for ODEs

- Equation with initial conditions

$$\frac{d^2y}{dt^2} + y = 1, \qquad y(0) = y'(0) = 0$$

- Laplace transform is linear

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(1)$$

- Apply derivative formula

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{1}{s}$$

- Rearrange

$$\mathcal{L}(y) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

- Take the inverse

$$y = 1 - \cos t$$

Laplace Transform in PDEs

Laplace transform in two variables (always taken with respect to time variable, t):

$$\mathcal{L}\{u(x,t)\} = U(x,s) = \int_0^{\infty} e^{-st} \frac{du}{dt}\, dt$$

Inverse laplace of a 2 dimensional PDE:

$$\mathcal{L}^{-1}\{U(x,s)\} = u(x,t)$$

Can be used for any dimension PDE:

$$\mathcal{L}\{u(x,y,z,t)\} = U(x,y,z,s)$$

The Transform reduces dimension by “1”:

- ODEs reduce to algebraic equations
- PDEs reduce to either an ODE (if original equation dimension 2) or another PDE (if original equation dimension >2)

Consider the case where:

$u_x+u_t=t$ with $u(x,0)=0$ and $u(0,t)=t^2$ and

Taking the Laplace of the initial equation leaves $U_x+U=1/s^2$ (note that the partials with respect to "x" do not disappear) with boundary condition $U(0,s)=2/s^3$

Solving this as an ODE of variable x, $U(x,s)=c(s)e^{-x}+1/s^2$

Plugging in B.C., $2/s^3=c(s)+1/s^2$ so $c(s)=2/s^3-1/s^2$

$U(x,s)=(2/s^3-1/s^2)e^{-x}+1/s^2$

Now, we can use the inverse Laplace Transform with respect to s to find

$u(x,t)=t^2e^{-x}-te^{-x}+t$

Example Solutions

Diffusion Equation

$u_t = ku_{xx}$ in $(0,l)$

Initial Conditions:

$u(0,t) = u(l,t) = 1, \quad u(x,0) = 1 + \sin(\pi x/l)$

Using $af(t) + bg(t) \rightarrow aF(s) + bG(s)$

and $df/dt \rightarrow sF(s) - f(0)$

and noting that the partials with respect to x commute with the transforms with respect to t, the Laplace transform U(x,s) satisfies

$sU(x,s) - u(x,0) = kU_{xx}(x,s)$

With $e^{at} \rightarrow 1/(s-a)$ and a=0,

the boundary conditions become U(0,s) = U(l,s) = 1/s.

So we have an ODE in the variable x together with some boundary conditions. The solution is then:

$U(x,s) = 1/s + (1/(s+k\pi^2/l^2))\sin(\pi x/l)$

Therefore, when we invert the transform, using the Laplace table:

$u(x,t) = 1 + e^{-k\pi^2 t/l^2}\sin(\pi x/l)$

Wave Equation

$u_{tt} = c^2 u_{xx}$ in $0 < x < \infty$

Initial Conditions:

$u(0,t) = f(t)$, $u(x,0) = u_t(x,0) = 0$

For $x \to \infty$, we assume that $u(x,t) \to 0$. Because the initial conditions vanish, the Laplace transform satisfies

$s^2 U = c^2 U_{xx}$

$U(0,s) = F(s)$

Solving this ODE, we get

$U(x,s) = a(s)e^{-sx/c} + b(s)e^{sx/c}$

Where a(s) and b(s) are to be determined.

From the assumed property of u, we expect that $U(x,s) \to 0$ as $x \to \infty$.

Therefore, $b(s) = 0$. Hence, $U(x,s) = F(s)\, e^{-sx/c}$. Now we use

$H(t-b)f(t-b) \to e^{-bs}F(s)$

To get

$u(x,t) = H(t - x/c)f(t - x/c)$.

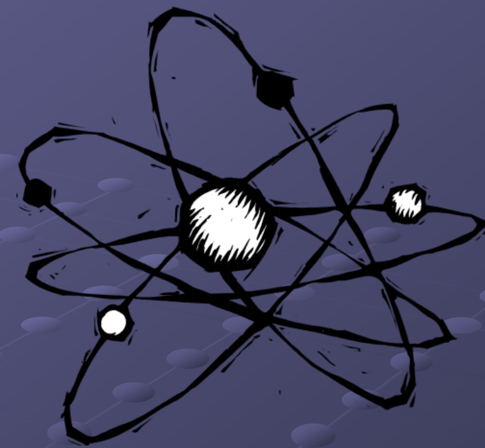
Real-Life Applications

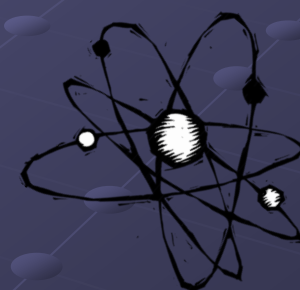
- Semiconductor mobility
- Call completion in wireless networks
- Vehicle vibrations on compressed rails
- Behavior of magnetic and electric fields above the atmosphere




Ex. Semiconductor Mobility

- Motivation
  - semiconductors are commonly made with superlattices having layers of differing compositions
  - need to determine properties of carriers in each layer
    - concentration of electrons and holes
    - mobility of electrons and holes
  - conductivity tensor can be related to Laplace transform of electron and hole densities





Notation

- R = ratio of induced electric field to the product of the current density and the applied magnetic field
- ρ = electrical resistance
- H = magnetic field
- J = current density
- E = applied electric field
- n = concentration of electrons
- *u* = mobility

$$E_x = \rho J_x - RHJ_y$$

$$E_y = RHJ_x + \rho J_y$$

Equation Manipulation

$$\rho = \frac{1}{ne\mu},$$

$$R = -\frac{1}{ne}$$

and

$$\sigma_{xx} = \frac{ne\mu}{[1+(\mu H)]^2};$$

$$\sigma_{xy} = \frac{-ne\mu^2 H}{[1+(\mu H)]^2}.$$

→

$$J_x = \sigma_{xx}E_x + \sigma_{xy}E_y,$$

$$J_y = -\sigma_{xy}E_x + \sigma_{xx}E_y,$$

Assuming a continuous mobility distribution and that $S^{+}(\mu)=k^{+}e\mu p_{\mu}$, $S^{-}(\mu)=k^{-}e\mu n_{\mu}$, it follows:

$$\sigma_{xx}=\sum_{\mu}\frac{(n_{\mu}+p_{\mu})e\mu}{[1+(\mu H)]^{2}},$$

$$\sigma_{xy}=\sum_{\mu}\frac{(p_{\mu}-n_{\mu})(e\mu^{2}H)}{[1+(\mu H)]^{2}}$$

$$\sigma_{xx}(H)=\int_{0}^{\infty}\frac{S^{+}(\mu)+S^{-}(\mu)}{1+(\mu H)^{2}}d\mu,$$

$$\sigma_{xy}(H)=\int_{0}^{\infty}\frac{[S^{+}(\mu)-S^{-}(\mu)]\mu H}{1+(\mu H)^{2}}d\mu.$$

Applying the Laplace Transform

$$\int_0^\infty e^{-yt}\sin(xt)\,dt = \frac{x}{x^2+y^2},$$

$$\int_0^\infty e^{-yt}\cos(xt)\,dt = \frac{y}{x^2+y^2}.$$

$$H\sigma_{xx} = \int_0^\infty e^{-yt}\left[\int_0^\infty (S^+ + S^-)\cos(xt)\,dx\right]dt$$

$$= \int_0^\infty e^{-yt}\left[\int_0^\infty ex(p_x + n_x)\cos(xt)\,dx\right]dt,$$

$$H\sigma_{xy} = \int_0^\infty e^{-yt}\left[\int_0^\infty (S^+ - S^-)\sin(xt)\,dx\right]dt$$

$$= \int_0^\infty e^{-yt}\left[\int_0^\infty ex(p_x - n_x)\sin(xt)\,dx\right]dt.$$

Source

Johnson, William B. Transform method for semiconductor mobility, Journal of Applied Physics 99 (2006).